Solutions for week 1 and 2

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Problem 1

Show that the 4D volumn element is invariant under Lorentz transformation:

$$d^4x = d^4x' \tag{1}$$

Solution:

$$d^4x = \det(\frac{\partial x^\mu}{\partial x'^\nu})d^4x' \tag{2}$$

$\frac{\partial x^\mu}{\partial x'^\nu}$ is a Lorentz transformation thus $\det(\frac{\partial x^\mu}{\partial x'^\nu}) = |\Lambda^\mu{}_\nu| = 1$, since $\Lambda^\mu{}_\nu \in SO(1,3)$ and $\Lambda^\mu{}_\nu g_{\mu\mu'} \Lambda^{\mu'}{}_{\nu'} = g_{\nu'\nu'}$.

Problem 2

Show that under a Lorentz transformation,

$$\frac{d^3k}{2\omega_k} \to \frac{d^3k'}{2\omega'_k} \tag{3}$$

namely, it's also Lorentz invariant.

Solution:

Notice

$$\begin{aligned} & \int \mathrm{d}^4k f(k)\, \delta\left(k^2 - m^2\right)\Big|_{k_0>0} \\ = & \int \mathrm{d}^3k\ \mathrm{d}k_0 f(k)\, \delta\left(k_0^2 - \boldsymbol{k}^2 - m^2\right)\Big|_{k_0>0} \\ = & \int \mathrm{d}^3k\ \mathrm{d}k_0 f(k_0, \boldsymbol{k})\, \frac{\delta\left(k_0 - \sqrt{\boldsymbol{k}^2 + m^2}\right)}{2\sqrt{\boldsymbol{k}^2 + m^2}}\Bigg|_{k_0>0} \\ = & \int \frac{\mathrm{d}^3k}{2\omega_k} f\left(\omega_k, \boldsymbol{k}\right) \end{aligned} \tag{4}$$

with $m = \sqrt{\omega_k^2 - \boldsymbol{k}^2}$. As $\delta(k^2 - m^2)|_{k_0>0} = \delta\left(k^2 - m^2\right)\theta\left(\omega_k\right)$ is Lorentz invariant, the identity above holds for any Lorentz invariant fuction $f(k)$ meaning

$$\begin{aligned} & \int \mathrm{d}^4 k f(k)\, \delta\left(k^2 - m^2\right)\big|_{k_0>0} \\ = & \int \mathrm{d}^4 k' f(k')\, \delta\left(k'^2 - m^2\right)\big|_{k_0>0} \\ = & \int \frac{\mathrm{d}^3 k}{2\omega_k} f\left(\omega_k, \boldsymbol{k}\right) \\ = & \int \frac{\mathrm{d}^3 k'}{2\omega_{k'}} f\left(\omega_k', \boldsymbol{k}'\right) \end{aligned} \tag{5}$$

So $\frac{d^3k}{2\omega_k}$ is Lorentz invariant.

Problem 3

Show that the action for a free Klein-Gordon field is invariant for Lorentz transformation Λ, where $\det(\Lambda) = 1$.

Solution:

For Klein-Gordon field

$$\begin{aligned} S &= \int \mathrm{d}^4 x \mathcal{L} \\ &= \int \mathrm{d}^4 x \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2\right) \\ &= \int \mathrm{d}^4 x' \det\left(\frac{\partial x^\mu}{\partial x^\nu}\right)\left(\frac{1}{2} g^{\mu\lambda} \Lambda^\nu{}_\mu \Lambda^\rho{}_\lambda \partial'_\nu \phi \partial'_\rho \phi - \frac{1}{2} m^2 \phi^2(x')\right) \\ &= \int \mathrm{d}^4 x' \left(\frac{1}{2}(\partial'_\nu \phi)^2 - \frac{1}{2} m^2 \phi^2\right). \end{aligned} \tag{6}$$

Problem 4

Explain why we don't need a linear term of ϕ in the Lagrangian for a free Klein-Gordon field.

Solution:

If there is a linear term in Lagrangian of a free Klein-Gordon field,

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + a\phi \\ &= \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \left(\phi - \frac{a}{m^2}\right)^2 + \frac{a^2}{2m^2} \\ &= \frac{1}{2}\partial_\mu \phi' \partial^\mu \phi' - \frac{1}{2} m^2 \phi'^2 + \frac{a^2}{2m^2} \end{aligned} \tag{7}$$

where $\phi' = \phi - \frac{a}{m^2}$. So this Lagrangian is equivalent to a one without a linear term.

Problem 5

In d space-time dimension, what is the dimension of the following Lagrangian? What is the dimension of a_n?

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \sum_{n=2}^{\infty} a_n\phi^n \tag{8}$$

Solution:

$$S = \int d^d x\mathcal{L} \tag{9}$$

and $[S] = 0$, then $[\mathcal{L}] = M^d \Rightarrow [\phi] \to \frac{d-2}{2} \Rightarrow [a_n] \to d - n\frac{d-2}{2}$

Problem 6

A space-time translation operator $T(a)$ is defined as $T(a) = \exp(ia^\mu P_\mu)$, where P_μ is the momentum operator. Show that $T(a)$ is unitary. A scalar field transforms under $T(a)$ as $T(a)^\dagger\phi(x)T(a) = \phi(x-a)$

Solution:

$$T(a)^\dagger T(a) = \exp(-ia^\mu P_\mu)\exp(ia^\mu P_\mu) = I \tag{10}$$

Part (a)

- Let a_μ be infinitesimal. Derive an expression for $[P_\mu, \phi(x)]$.

Solution:

As $a^\mu \to 0$, $T(a) = 1 + ia_\mu P^\mu + O(a^2)$ and $\phi(x-a) = \phi(x) - a^\mu\partial_\mu\phi(x) + O(a^2)$.

$$\begin{aligned} T(a)^\dagger\phi(x)T(a) &= \phi(x) - ia^\mu[P_\mu, \phi(x)] + O(a^2) \\ &= \phi(x) - a^\mu\partial_\mu\phi(x) + O(a^2) \end{aligned} \tag{11}$$

So

$$\partial^\mu\phi(x) = i[P^\mu, \phi(x)] \tag{12}$$

Part (b)

- Show that the time component of your result is equivalent to the Heisenberg equation of motion $\dot{\phi} = i[H, \phi(x)]$.

Solution:

As $P^\mu = (H, \boldsymbol{P})$, the time component of the result is $\partial_0\phi(x) = i[H, \phi(x)] \Rightarrow \dot{\phi} = i[H, \phi(x)]$

Part (c)

- For a free field, use the Heisenberg equation to derive the Klein-Gordon equation.

0.0.1 Solution:

$$\begin{aligned}\dot{\phi} &= i[H,\phi] = i\left[\frac{1}{2}\int \mathrm{d}^3x'\left(\pi^2+(\nabla\phi)^2+m^2\phi^2\right),\phi\right]\\ &= \int d^3x'\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x'})\pi(\boldsymbol{x'},t)\\ &= \pi(\boldsymbol{x},t)\end{aligned} \tag{13}$$

$$\begin{aligned}\dot{\pi} &= i[H,\pi] = i\left[\frac{1}{2}\int \mathrm{d}^3x'\left(\pi^2+(\nabla\phi)^2+m^2\phi^2\right),\pi\right]\\ &= \int d^3x'\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x'})(\nabla^2-m^2)\phi(\boldsymbol{x'},t)\\ &= (\nabla^2-m^2)\phi(\boldsymbol{x},t)\end{aligned} \tag{14}$$

Also notice $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$, then there is the Klein-Gordon equation $\partial_t^2\phi = (\nabla^2-m^2)\phi(\boldsymbol{x},t)$. And in the derivation about the following facts are used.

$$[\phi(\boldsymbol{x'},t),\phi(\boldsymbol{x},t)] = [\pi(\boldsymbol{x'},t),\pi(\boldsymbol{x},t)] = 0 \tag{15}$$

$$[\nabla\phi(\boldsymbol{x'},t),\phi(\boldsymbol{x},t)] = \nabla_{\boldsymbol{x'}}[\phi(\boldsymbol{x'},t),\phi(\boldsymbol{x},t)] = 0 \tag{16}$$

$$[\phi(\boldsymbol{x},t),\pi(\boldsymbol{y},t)] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{17}$$

$$\begin{aligned}\left[\pi(\boldsymbol{x'},t)^2,\phi(\boldsymbol{x},t)\right] &= \pi(\boldsymbol{x'},t)[\pi(\boldsymbol{x'},t),\phi(\boldsymbol{x},t)] + [\pi(\boldsymbol{x'},t),\phi(\boldsymbol{x},t)]\pi(\boldsymbol{x'},t)\\ &= -2i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x'})\pi(\boldsymbol{x'},t)\end{aligned} \tag{18}$$

$$\begin{aligned}\left[\phi(\boldsymbol{x'},t)^2,\pi(\boldsymbol{x},t)\right] &= \phi(\boldsymbol{x'},t)[\phi(\boldsymbol{x'},t),\pi(\boldsymbol{x},t)] + [\phi(\boldsymbol{x'},t),\pi(\boldsymbol{x},t)]\phi(\boldsymbol{x'},t)\\ &= 2i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x'})\phi(\boldsymbol{x'},t)\end{aligned} \tag{19}$$

$$\begin{aligned}\left[(\nabla\phi(\boldsymbol{x'},t))^2,\pi(\boldsymbol{x},t)\right] &= \nabla\phi(\boldsymbol{x'},t)[\nabla\phi(\boldsymbol{x'},t),\pi(\boldsymbol{x},t)] + [\nabla\phi(\boldsymbol{x'},t),\pi(\boldsymbol{x},t)]\nabla\phi(\mathbf{x}',t)\\ &= 2i\nabla\phi(\boldsymbol{x'},t)\nabla_{\boldsymbol{x'}}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x'})\\ &= -2i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x'})\nabla^2\phi(\boldsymbol{x'},t)\end{aligned} \tag{20}$$

Part (d)

Define a spatial momentum operator

$$\boldsymbol{P} = -\int d^3x\pi(x)\boldsymbol{\nabla}\phi(x) \tag{21}$$

Use the canonical commutation relations to show that $\boldsymbol{P}$ obeys the relation you derived in part (a).

Solution:

$$\begin{aligned}[\boldsymbol{P}(t),\phi(\boldsymbol{x},t)] &= -\int d^3\boldsymbol{y}[\pi(\boldsymbol{y},t)\nabla\phi(\boldsymbol{y},t),\phi(\boldsymbol{x},t)]\\ &= -\int d^3\boldsymbol{y}[\pi(\boldsymbol{y},t),\phi(\boldsymbol{x},t)]\nabla\phi(\boldsymbol{y},t)\\ &= i\int d^3\boldsymbol{y}\delta^{(3)}(\boldsymbol{x}-\mathbf{y})\nabla\phi(\boldsymbol{y},t)\\ &= i\nabla\phi(\boldsymbol{x},t)\end{aligned} \tag{22}$$

Part (e)

Express $\boldsymbol{P}$ in terms of $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$.

Solution:

Let's denote $\int \widetilde{dk} = \int \frac{d^3k}{(2\pi)^3 2\omega_{\boldsymbol{k}}}$. As $\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$ and

$$\phi(x) = \int \tilde{dk}(a_{\boldsymbol{k}} e^{ikx} + a_{\boldsymbol{k}}^\dagger e^{-ikx}) \tag{23}$$

so

$$\pi(x) = \int \tilde{dk}(i\omega_{\boldsymbol{k}} a_{\boldsymbol{k}} e^{ikx} - i\omega_{\boldsymbol{k}} a_{\boldsymbol{k}}^\dagger e^{-ikx}) \tag{24}$$

and

$$\nabla\phi(x) = \int \tilde{dk}(-i\boldsymbol{k} a_{\boldsymbol{k}} e^{ikx} + i\boldsymbol{k} a_{\boldsymbol{k}}^\dagger e^{-ikx}) \tag{25}$$

then

$$\begin{aligned}
P &= -\int d^3x \dot{\phi}(x) \nabla\phi(x) \\
&= -\int d^3x \left[\int \widetilde{dk}(i\omega_{\boldsymbol{k}} a_{\boldsymbol{k}} e^{ikx} - i\omega_{\boldsymbol{k}} a_{\boldsymbol{k}}^\dagger e^{-ikx})\right]\left[\int \widetilde{dk} i\mathbf{k}(-i\boldsymbol{k} a_{\boldsymbol{k}} e^{ikx} + i\boldsymbol{k} a_{\boldsymbol{k}}^\dagger e^{-ikx})\right] \\
&= -\int d^3x \widetilde{dk}\widetilde{dk'} \omega_{\boldsymbol{k}} \mathbf{k}' \left[a_{\boldsymbol{k}} e^{ikx} - a_{\boldsymbol{k}}^\dagger e^{-ikx}\right]\left[a_{\boldsymbol{k'}} e^{ik'x} - a_{\boldsymbol{k'}}^\dagger e^{-ik'x}\right] \\
&= -\int d^3x \widetilde{dk}\widetilde{dk'} \omega_{\boldsymbol{k}} \mathbf{k}' \left[a_{\boldsymbol{k}} a_{\boldsymbol{k'}} e^{i(k+k')x} - a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k'}} e^{i(k'-k)x} - a_{\boldsymbol{k}} a_{\boldsymbol{k'}}^\dagger e^{i(k-k')x} + a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k'}}^\dagger e^{-i(k+k')x}\right] \\
&= -(2\pi)^3 \int \widetilde{dk}\widetilde{dk'} \omega_{\boldsymbol{k}} \mathbf{k}' \left[(a_{\boldsymbol{k}} a_{\boldsymbol{k'}} e^{i(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k'}})t} + a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k'}}^\dagger e^{-i(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k'}})t})\delta^{(3)}(\boldsymbol{k}+\boldsymbol{k'})\right. \\
&\quad \left. -(a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k'}} e^{i(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k'}})t} + a_{\boldsymbol{k}} a_{\boldsymbol{k'}}^\dagger e^{-i(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k'}})t})\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k'})\right] \\
&= -\frac{1}{2}\int \widetilde{dk}\mathbf{k}\left[(a_{\boldsymbol{k}} a_{-\boldsymbol{k}} e^{i2\omega_{\boldsymbol{k}}t} + a_{\boldsymbol{k}}^\dagger a_{-\boldsymbol{k}}^\dagger e^{-i2\omega_{\boldsymbol{k}}t}) - (a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + a_{\boldsymbol{k}} a_{\boldsymbol{k}}^\dagger)\right] \\
&= \frac{1}{2}\int \widetilde{dk}\mathbf{k}(a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + a_{\boldsymbol{k}} a_{\boldsymbol{k}}^\dagger) \\
&= \int \widetilde{dk}\mathbf{k} a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}
\end{aligned} \tag{26}$$

In the last two lines all odd terms (thanks to the k in front) vanish when integrating over all space.

Problem 7

The time-ordered product of two fields, $A(x)$ and $B(y)$, is defined by

$$T[A(x)B(y)] = \theta\left(x^0 - y^0\right) A(x)B(y) + \theta\left(y^0 - x^0\right) B(y)A(x) \tag{27}$$

Using only the field equation and the equal time commutation relations, show that, for a free scalar field of mass m,

$$\left(\partial_x^2 + m^2\right)\langle 0|T[\phi(x)\phi(y)]|0\rangle = c\delta^{(4)}(x-y) \tag{28}$$

Find the constant c.

Solution:

Notice $\exp(ia^\mu P_\mu)\phi(x)\exp(-ia^\mu P_\mu) = \phi(x+a)$, and it can be used to translate y to the origin.

$$\langle 0|T[\phi(x)\phi(y)]|0\rangle = \langle 0|T[e^{iy^\mu P_\mu}\phi(x-y)e^{-iy^\mu P_\mu}e^{iy^\mu P_\mu}\phi(0)e^{-iy^\mu P_\mu}]|0\rangle = \langle 0|T[\phi(x-y)\phi(0)]|0\rangle \tag{29}$$

For simplicity let's consider $\langle 0|T[\phi(x)\phi(0)]|0\rangle$.

From the Klein-Gordon equation $(\partial^2+m^2)\phi(x)=0$ and $\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$, then

$$\begin{aligned}
&(\partial^2+m^2)T[\phi(x)\phi(0)] = (\partial_t^2 - \nabla^2 + m^2)\left(\theta\left(x^0\right)\phi(x)\phi(0) + \theta\left(-x^0\right)\phi(0)\phi(x)\right) \\
&= \partial_t^2\left(\theta\left(x^0\right)\phi(x)\phi(0) + \theta\left(-x^0\right)\phi(0)\phi(x)\right) \\
&\quad + (-\nabla^2+m^2)\left(\theta\left(x^0\right)\phi(x)\phi(0) + \theta\left(-x^0\right)\phi(0)\phi(x)\right) \\
&= \partial_t\left(\delta\left(x^0\right)\phi(x)\phi(0) - \delta\left(x^0\right)\phi(0)\phi(x) + \theta\left(x^0\right)\pi(x)\phi(0) + \theta\left(-x^0\right)\phi(0)\pi(x)\right) \\
&\quad + (-\nabla^2+m^2)\left(\theta\left(x^0\right)\phi(x)\phi(0) + \theta\left(-x^0\right)\phi(0)\phi(x)\right) \\
&= \partial_t\left(\theta\left(x^0\right)\pi(x)\phi(0) + \theta\left(-x^0\right)\phi(0)\pi(x)\right) \\
&\quad + (-\nabla^2+m^2)\left(\theta\left(x^0\right)\phi(x)\phi(0) + \theta\left(-x^0\right)\phi(0)\phi(x)\right) \\
&= \left(\delta\left(x^0\right)\pi(x)\phi(0) - \delta\left(x^0\right)\phi(0)\pi(x)\right) + \left(\theta\left(x^0\right)\dot{\pi}(x)\phi(0) + \theta\left(-x^0\right)\phi(0)\dot{\pi}(x)\right) \\
&\quad + (-\nabla^2+m^2)\left(\theta\left(x^0\right)\phi(x)\phi(0) + \theta\left(-x^0\right)\phi(0)\phi(x)\right) \\
&= \delta\left(x^0\right)[\pi(x),\phi(0)] + T\left[(\partial^2+m^2)\phi(x)\phi(0)\right] \\
&= -i\delta^{(4)}(x)
\end{aligned} \tag{30}$$

which is $(\partial^2+m^2)T[\phi(x)\phi(0)] = -i\delta^{(4)}(x)$, then $\langle 0|(\partial^2+m^2)T[\phi(x)\phi(0)]|0\rangle = -i\delta^{(4)}(x)$. This implies

$$\left(\partial_x^2+m^2\right)\langle 0|T[\phi(x)\phi(y)]|0\rangle = -i\delta^{(4)}(x-y) \tag{31}$$

which means $c=-i$.

Problem 8

Considering the following Lagrangian:

$$\mathcal{L} = i\psi^*\partial_0\psi + b\nabla\psi^*\cdot\nabla\psi \tag{32}$$

where ψ is a complex scalar field and b is a real constant.

Part (a)

•Find the Euler–Lagrange equations.

Solution:

From $\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$, there are

$$\begin{aligned} i\partial_0\psi^* &= -b\nabla^2\psi^* \\ i\partial_0\psi &= b\nabla^2\psi \end{aligned} \tag{33}$$

Part(b)

•Find the plane-wave solutions, those for which ψ is of the form $\psi = e^{-i\omega t + i\boldsymbol{p}\cdot\boldsymbol{x}}$, and find ω as a function of $\boldsymbol{p}$.

Solution:

Expand the classical scalar field

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} e^{-ipx}\psi(p) \tag{34}$$

then

$$(\omega + b\boldsymbol{p}^2)\psi(p) = 0 \tag{35}$$

$$\omega = -b\boldsymbol{p}^2 \tag{36}$$

Part (c)

•Although this theory is not Lorentz-invariant, it is invariant under spacetime translations and the internal symmetry transformation

$$\psi \to e^{-i\alpha}\psi, \quad \psi^* \to e^{i\alpha}\psi^* \tag{37}$$

Thus it possesses a conserved energy, a conserved linear momentum, and a conserved charge associated with the internal symmetry. Find these quantities as integrals of the fields and their derivatives. Fix the sign of b by demanding the energy be bounded below.

Solution:

From $P_\mu = \int d^3x \mathcal{T}_{0\mu}$ and $Q = \int d^3x J_0$, where $\mathcal{T}_{\mu\nu} = \sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\partial_\nu\phi_n - g_{\mu\nu}\mathcal{L}$ and $J_\mu = \sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\frac{\delta\phi_n}{\delta\alpha}$,

$$\begin{aligned} H &= -b\int d^3x \nabla\psi^*\cdot\nabla\psi \\ \boldsymbol{P} &= -\int d^3x i\psi^*\nabla\psi \\ Q &= \int d^3x \psi^*\psi \end{aligned} \tag{38}$$

Thus $b < 0$.

Part (d)

•Canonically quantize the theory. Identify appropriately normalized coefficients in the expansion of the fields in terms of plane wave solutions with annihilation and/or creation operators. Write the energy, linear momentum and internal-symmetry charge in terms of these operators.

Solution:

Assume

$$\psi(\boldsymbol{x}, t) = \int \frac{d^3p}{(2\pi)^3}\left(u(\boldsymbol{p})a_{\boldsymbol{p}}e^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} + v(\boldsymbol{p})b^\dagger_{\boldsymbol{p}}e^{-i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{39}$$

then

$$\psi^*(\boldsymbol{x}, t) = \int \frac{d^3p}{(2\pi)^3}\left(v^*(\boldsymbol{p})b_{\boldsymbol{p}}e^{i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} + u^*(\boldsymbol{p})a^\dagger_{\boldsymbol{p}}e^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{40}$$

where $\omega_{\boldsymbol{p}} = -b\boldsymbol{p}^2$.

When imposing

$$\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p}') \tag{41}$$

and

$$\left[b_{\mathbf{p}}, b_{\mathbf{p}'}^{\dagger}\right] = (2\pi)^3\delta^{(3)}\left(\mathbf{p}-\mathbf{p}'\right) \tag{42}$$

there are

$$\begin{gathered}[\psi(\mathbf{x},t),\psi(\mathbf{y},t)] = [\pi(\mathbf{x},t),\pi(\mathbf{y},t)] = 0\\ [\psi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\end{gathered} \tag{43}$$

where $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^*$, when $u(\boldsymbol{p})u^*(\boldsymbol{p}) - v(-\boldsymbol{p})v^*(-\boldsymbol{p}) = 1$. Moreover, this theory can't be bounded below if $v(p) \neq 0$, which tells that in this scalar field without Lorentz symmetry anti-particles do not show up. So the correct field operator show be

$$\psi(\boldsymbol{x},t) = \int \frac{d^3p}{(2\pi)^3} a_{\boldsymbol{p}} e^{-i\omega_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{44}$$

with

$$\left[a_{\mathbf{p}}, a_{\mathbf{p}'}^{\dagger}\right] = (2\pi)^3\delta^{(3)}\left(\mathbf{p}-\mathbf{p}'\right) \tag{45}$$

So

$$H = \int d^3p\omega_{\boldsymbol{p}} a_{\boldsymbol{p}}^{\dagger} a_{\boldsymbol{p}} \tag{46}$$

$$\boldsymbol{P} = \int d^3p\boldsymbol{p} a_{\boldsymbol{p}}^{\dagger} a_{\boldsymbol{p}} \tag{47}$$

$$Q = \int \mathrm{d}^3p a_{\boldsymbol{p}}^{\dagger} a_{\boldsymbol{p}} \tag{48}$$

Part (e)

• Find the equation of motion for the single particle state $|k\rangle$ and the two particle state $|k_1k_2\rangle$ in the Schrodinger Picture. What physical quantities do b and the internal symmetry charge correspond to?

Solution:

A one particle sate with proper normalization ($\langle\boldsymbol{p}|\boldsymbol{q}\rangle = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$) is $|\boldsymbol{k}\rangle = a_{\boldsymbol{k}}^{\dagger}|0\rangle$.

$$\begin{aligned}\frac{\partial}{\partial t}|\boldsymbol{k}\rangle &= \frac{\partial}{\partial t}a_{\boldsymbol{k}}^{\dagger}|0\rangle = -i[a^{\dagger}, H]|0\rangle\\ \Rightarrow i\frac{\partial}{\partial t}|\boldsymbol{k}\rangle &= \omega_{\boldsymbol{k}}|\boldsymbol{k}\rangle = -b\boldsymbol{k}^2|\boldsymbol{k}\rangle\end{aligned} \tag{49}$$

Similarly,

$$i\frac{\partial}{\partial t}\left|\mathbf{k}_1\mathbf{k}_2\right\rangle = -b\left(k_1^2+k_2^2\right)\left|\mathbf{k}_1\mathbf{k}_2\right\rangle \tag{50}$$

b corresponds to m^{-1} and internal symmetry charge corresponds to the number of particles.