第五、六周作业参考答案

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习题 1

证明关于泡利矩阵的如下公式：

$$\exp\left[-i\frac{\vec{\sigma}\cdot\hat{n}}{2}\theta\right]=\mathbf{1}\cos\frac{\theta}{2}-i\vec{\sigma}\cdot\hat{n}\sin\frac{\theta}{2} \tag{1}$$

$$\exp\left[\frac{\vec{\sigma}\cdot\hat{n}}{2}\phi\right]=\mathbf{1}\cosh\frac{\phi}{2}+\vec{\sigma}\cdot\hat{n}\sinh\frac{\phi}{2} \tag{2}$$

$$\left\{\sigma^i,\sigma^j\right\}=2\delta^{ij}\mathbf{1} \tag{3}$$

其中 $\hat{n}$ 是单位矢量，θ 和 ϕ 是实数。

解：

可以验证

$$\sigma^i\sigma^j=\delta^{ij}\mathbf{1}+i\epsilon^{ijk}\sigma^k \tag{4}$$

于是

$$\left\{\sigma^i,\sigma^j\right\}=2\delta^{ij}\mathbf{1} \tag{5}$$

同时

$$\begin{aligned}(\vec{\sigma}\cdot\hat{n})^2&=\hat{n}_i\sigma_i\hat{n}_j\sigma_j\\&=\hat{n}_i\hat{n}_j\left(\delta^{ij}\mathbf{1}+i\epsilon^{ijk}\sigma^k\right)\\&=\hat{n}_i\hat{n}_j\delta^{ij}\mathbf{1}\\&=\mathbf{1}\end{aligned} \tag{6}$$

所以

$$\begin{aligned}\exp\left[-i\frac{\vec{\sigma}\cdot\hat{n}}{2}\theta\right]&=\sum_{k=0}^{\infty}\frac{(-i)^k}{k!}\left(\frac{\vec{\sigma}\cdot\hat{n}}{2}\right)^k\theta^k\\&=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{\vec{\sigma}\cdot\hat{n}}{2}\right)^{2k}\theta^{2k}-i\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{\vec{\sigma}\cdot\hat{n}}{2}\right)^{2k+1}\theta^{2k+1}\\&=\mathbf{1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{\theta}{2}\right)^{2k}-i\vec{\sigma}\cdot\hat{n}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{\theta}{2}\right)^{2k+1}\\&=\mathbf{1}\cos\frac{\theta}{2}-i\vec{\sigma}\cdot\hat{n}\sin\frac{\theta}{2}.\end{aligned} \tag{7}$$

代入 $\theta = i\phi$ 得到

$$\exp\left[\frac{\vec{\sigma}\cdot\hat{n}}{2}\phi\right] = \mathbf{1}\cosh\frac{\phi}{2} + \vec{\sigma}\cdot\hat{n}\sinh\frac{\phi}{2} \tag{8}$$

习题 2

证明 $SU(2)$ 的旋量表示（自旋 1/2 表示）的复共轭表示是等价表示，即存在 S，使得

$$U_R(\vec{n},\theta) = SU_R^*(\vec{n},\theta)S^\dagger \tag{9}$$

其中

$$U_R(\vec{n},\theta) = \exp\left[-i\frac{\vec{\sigma}\cdot\hat{n}}{2}\theta\right] \tag{10}$$

找出 S。

解:

可以验证如下性质

$$\vec{\sigma}^* = -\sigma^2\vec{\sigma}\sigma^2 \tag{11}$$

则

$$\begin{aligned}\sigma^2 U_R(\vec{n},\theta)\sigma^2 &= \sum_{k=0}^{\infty}\frac{1}{k!}\sigma^2\left(-i\vec{\sigma}\cdot\hat{n}\right)^k\sigma^2\left(\frac{\theta}{2}\right)^k \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(i\vec{\sigma}^*\cdot\hat{n}\right)^k\left(\frac{\theta}{2}\right)^k \\ &= U_R(\vec{n},\theta)^*\end{aligned} \tag{12}$$

或者由习题 1 结论

$$\begin{aligned}\sigma^2 U_R(\vec{n},\theta)\sigma^2 &= \mathbf{1}\cos\frac{\theta}{2} - i\sigma^2(\vec{\sigma}\cdot\hat{n})\sigma^2\sin\frac{\theta}{2} \\ &= \mathbf{1}\cos\frac{\theta}{2} + i\vec{\sigma}^*\cdot\hat{n}\sin\frac{\theta}{2} \\ &= U_R(\vec{n},\theta)^*\end{aligned} \tag{13}$$

因此

$$S = \sigma^2$$

习题 3

验证泡利矩阵满足

$$\sigma^\mu_{\alpha\beta}\sigma_{\mu,\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} \tag{14}$$

其中 $\sigma^\mu = (\mathbf{1},\vec{\sigma})$ ，$\epsilon_{\alpha\beta}$ 是完全反对称的二阶张量，$\epsilon_{12} = 1$。

解:

注意到

$$\mathrm{Tr}\{\sigma^i\} = 0 \tag{15}$$

和

$$\frac{1}{2}\,\mathrm{Tr}\{\sigma^i\sigma^j\} = \delta^{ij} \tag{16}$$

对任意的二阶复矩阵，可写成

$$M = M^\mu \sigma_\mu \tag{17}$$

则

$$\begin{aligned} M_{ab} = M^\mu \sigma_{\mu,ab} &= \frac{1}{2}\,\mathrm{Tr}(M)\sigma_{0,ab} - \frac{1}{2}\,\mathrm{Tr}(M\sigma^i)\sigma_{i,ab} \\ &= \frac{1}{2}M_{cd}\delta_{dc}\delta_{ab} - \frac{1}{2}M_{cd}\sigma^i_{dc}\sigma_{i,ab} \\ \Rightarrow &\sigma^i_{dc}\sigma_{i,ab} = \delta_{ab}\delta_{dc} - 2\delta_{ac}\delta_{bd} \end{aligned} \tag{18}$$

所以

$$\sigma^\mu_{\alpha\beta}\sigma_{\mu,\gamma\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta} + \sigma^i_{\alpha\beta}\sigma_{i,\gamma\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta} + (\delta_{\alpha\beta}\delta_{\gamma\delta} - 2\delta_{\alpha\delta}\delta_{\beta\gamma}) = 2\left(\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}\right) = 2\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} \tag{19}$$

习题 4

从洛伦兹群的旋转生成元 $\mathbb{J}$ 和 boost 生成元 $\mathbb{K}$ 出发，定义一组新的生成元：

$$\mathbb{J}_+ = \frac{1}{2}(\mathbb{J} + i\mathbb{K}), \quad \mathbb{J}_- = \frac{1}{2}(\mathbb{J} - i\mathbb{K}), \tag{20}$$

计算 $\mathbb{J}_+$ 和 $\mathbb{J}_-$ 的自身及其相互对易关系。

解:

通过

$$\begin{aligned} \left[\mathbb{J}^i, \mathbb{J}^j\right] &= i\varepsilon^{ijk}\mathbb{J}^k \\ \left[\mathbb{J}^i, \mathbb{K}^j\right] &= i\varepsilon^{ijk}\mathbb{K}^k \\ \left[\mathbb{K}^i, \mathbb{K}^j\right] &= -i\varepsilon^{ijk}\mathbb{J}^k \end{aligned} \tag{21}$$

可得

$$\begin{aligned} \left[\mathbb{J}^i_\pm, \mathbb{J}^j_\pm\right] &= i\varepsilon^{ijk}\mathbb{J}^k_\pm \\ \left[\mathbb{J}^i_+, \mathbb{J}^j_-\right] &= 0 \end{aligned} \tag{22}$$

习题 5

计算如下 gamma 矩阵的迹：

$$\mathrm{Tr}[\not{a}\not{b}\not{c}], \quad \mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}], \quad \mathrm{Tr}\left[\not{a}\not{b}\not{c}\not{d}\gamma_5\right], \tag{23}$$

解：

由于

$$\begin{aligned}\operatorname{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\right] &= \operatorname{Tr}\left[\gamma_5\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\right] \\ &= -\operatorname{Tr}\left[\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma_5\right] \\ &= -\operatorname{Tr}\left[\gamma_5\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\right] \\ &= -\operatorname{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\right]\end{aligned} \tag{24}$$

则

$$\operatorname{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\right] = 0 \tag{25}$$

$$\operatorname{Tr}[\not a \not b \not c] = 0 \tag{26}$$

由于

$$\begin{aligned}\operatorname{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] &= 2\eta^{\mu\nu}\operatorname{Tr}\left[\gamma^\rho\gamma^\sigma\right] - \operatorname{Tr}\left[\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\right] \\ &= 2\eta^{\mu\nu}\operatorname{Tr}\left[\gamma^\rho\gamma^\sigma\right] - 2\eta^{\mu\rho}\operatorname{Tr}\left[\gamma^\nu\gamma^\sigma\right] + \operatorname{Tr}\left[\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\right] \\ &= 2\eta^{\mu\nu}\operatorname{Tr}\left[\gamma^\rho\gamma^\sigma\right] - 2\eta^{\mu\rho}\operatorname{Tr}\left[\gamma^\nu\gamma^\sigma\right] + 2\eta^{\mu\sigma}\operatorname{Tr}\left[\gamma^\nu\gamma^\rho\right] - \operatorname{Tr}\left[\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu\right] \\ &= 8\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) - \operatorname{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] \\ &= 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)\end{aligned} \tag{27}$$

则

$$\operatorname{Tr}[\not a \not b \not c \not d] = 4[(ad)(bc) - (ac)(bd) + (ab)(dc)]. \tag{28}$$

由于

$$\operatorname{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5\right] = -4i\epsilon^{\mu\nu\rho\sigma} \tag{29}$$

$$\operatorname{Tr}\left[\not a \not b \not c \not d \gamma_5\right] = -4i\epsilon^{\mu\nu\rho\sigma}a_\mu b_\nu c_\rho d_\sigma \tag{30}$$

习题 6

分别计算如下外尔场和狄拉克场的能量动量张量：

$$\mathcal{L} = i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R \quad , \quad \mathcal{L} = \bar\psi i\gamma^\mu\partial_\mu\psi - m\bar\psi\psi \tag{31}$$

解：

对于外尔场的能量动量张量：

$$\begin{aligned}T^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi_R\right)}\partial^\nu\psi_R + \partial^\nu\psi_R^\dagger\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi_R^\dagger\right)} - \mathcal{L}g^{\mu\nu} \\ &= \psi_R^\dagger i\sigma^\mu\partial^\nu\psi_R - g^{\mu\nu}i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R\end{aligned} \tag{32}$$

对于狄拉克场的能量动量张量：

$$\begin{aligned}T^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\partial^\nu\psi + \partial^\nu\bar\psi\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\bar\psi\right)} - \mathcal{L}g^{\mu\nu} \\ &= \bar\psi i\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\bar\psi i\gamma^\mu\partial_\mu\psi + g^{\mu\nu}m\bar\psi\psi\end{aligned} \tag{33}$$

习题 7

内部对称性和时空对称性之外的一个有趣推广是超对称。考虑如下包含复标量场和外尔旋量场的拉氏量：

$$\mathcal{L}_0 = \partial_\mu \phi^* \partial^\mu \phi + i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + F^* F \tag{34}$$

其中 F 是辅助复标量场，其运动方程为 $F = 0$。

问题 (a)

- 证明这个拉氏量在如下超对称变换下不变：

$$\delta\phi = -i\eta^T \sigma^2 \psi_L, \quad \delta\psi_L = \eta F + \sigma^\mu \partial_\mu \phi \sigma^2 \eta^*, \quad \delta F = -i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L \tag{35}$$

其中 $\eta_a, a = 1, 2$ 是一个二分量的复格拉斯曼数，即满足反对易条件的数 $\eta_a \eta_b = -\eta_b \eta_a$。

解：

可验证

$$\delta\left(\partial_\mu \phi^* \partial^\mu \phi\right) = \mathrm{i}\left(\partial_\mu \psi_L^\dagger \sigma^2 \eta^*\right) \partial^\mu \phi + \left(\partial_\mu \phi^*\right)\left(-\mathrm{i}\eta^T \sigma^2 \partial^\mu \psi_L\right) \tag{36}$$

$$\begin{aligned}
\delta\left(\psi_L^\dagger \mathrm{i}\bar{\sigma}^\mu \partial_\mu \psi_L\right) =& \left(F^* \eta^\dagger + \eta^T \sigma^2 \sigma^\nu \partial_\nu \phi^*\right) \mathrm{i}\bar{\sigma}^\mu \partial_\mu \psi_L + \psi_L^\dagger \mathrm{i}\bar{\sigma}^\mu \left(\eta \partial_\mu F + \sigma^\nu \sigma^2 \eta^* \partial_\mu \partial_\nu \phi\right) \\
=& \mathrm{i}F^* \eta^\dagger \bar{\sigma}^2 \partial_\mu \psi_L + \mathrm{i}\partial_\mu \left[\eta^T \sigma^2 \sigma^\nu \bar{\sigma}^\mu \left(\partial_\nu \phi^*\right) \psi_L\right] - \mathrm{i}\eta^T \sigma^2 \sigma^\nu \bar{\sigma}^\mu \left(\partial_\nu \partial_\mu \phi^*\right) \psi_L \\
&+ \mathrm{i}\psi_L^\dagger \bar{\sigma}^\mu \eta \partial_\mu F + \mathrm{i}\psi_L^\dagger \bar{\sigma}^\mu \sigma^\nu \sigma^2 \eta^* \partial_\mu \partial_\nu \phi \\
=& \mathrm{i}F^* \eta^\dagger \bar{\sigma}^2 \partial_\mu \psi_L + \mathrm{i}\partial_\mu \left[\eta^T \sigma^2 \sigma^\nu \bar{\sigma}^\mu \left(\partial_\nu \phi^*\right) \psi_L\right] - \mathrm{i}\eta^T \sigma^2 \left(\partial^2 \phi^*\right) \psi_L \\
&+ \mathrm{i}\psi_L^\dagger \bar{\sigma}^\mu \eta \partial_\mu F + \mathrm{i}\psi_L^\dagger \sigma^2 \eta^* \partial^2 \phi
\end{aligned} \tag{37}$$

$$\delta\left(F^* F\right) = \mathrm{i}\left(\partial_\mu \psi_L^\dagger\right) \bar{\sigma}^\mu \eta F - \mathrm{i}F^* \eta^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L \tag{38}$$

于是得到

$$\delta\mathcal{L} = \mathrm{i}\partial_\mu \left[\psi_L^\dagger \sigma^2 \eta^* \partial^\mu \phi + \psi_L^\dagger \bar{\sigma}^\mu \eta F + \phi^* \eta^T \sigma^2 \left(\sigma^\mu \sigma^\nu \sigma_\nu - \partial^\mu\right) \psi_L\right] \tag{39}$$

为一个全导数项。

问题 (b)

- 上述拉氏量对应的是自由理论。可以加入场的更高阶项引入相互作用。证明如下相互作用拉氏量仍然在超对称变换下不变：

$$\mathcal{L} = \mathcal{L}_0 + \left(F \frac{\partial W[\phi]}{\partial \phi} + \frac{i}{2} \frac{\partial^2 W[\phi]}{\partial \phi^2} \psi_L^T \sigma^2 \psi_L + \text{ c.c. }\right) \tag{40}$$

其中 $W[\phi]$ 是一个关于 ϕ 的任意的复函数。对于最简单例子 $W = g\phi^3/3$，计算 ϕ 和 ψ_L 所满足的场方程。

解：

$$
\begin{aligned}
\delta\mathcal{L}_{int} =&\delta\left(F\frac{\partial W[\phi]}{\partial\phi}+\frac{i}{2}\frac{\partial^2 W[\phi]}{\partial\phi^2}\psi_L^T\sigma^2\psi_L+\text{ c.c. }\right)\\
=&-i\eta^\dagger\bar{\sigma}\cdot\partial\psi_L\frac{\partial W}{\partial\phi}+F\frac{\partial^2 W}{\partial\phi^2}\left(-i\eta^T\sigma^2\psi_L\right)+\frac{i}{2}\frac{\partial^2 W}{\partial\phi^2}\psi_L^T\sigma^2\left(\eta F+\sigma\cdot\partial\phi\sigma^2\eta^*\right)\\
&+\frac{i}{2}\frac{\partial^2 W}{\partial\phi^2}(\eta F+\sigma\cdot\partial\phi\sigma^2\eta^*)^T\sigma^2\psi_L+\frac{i}{2}\frac{\partial^3 W}{\partial\phi^3}\left(-i\eta^T\sigma^2\psi_L\right)\left(\psi_L^T\sigma^2\psi_L\right)+c.c.\\
=&-i\left(\eta^\dagger\bar{\sigma}\cdot\partial\psi_L\frac{\partial W}{\partial\phi}-\frac{\partial^2 W}{\partial\phi^2}\eta^\dagger(\sigma^2)^T\sigma^T\cdot\partial\phi\sigma^2\psi_L\right)+\frac{i}{2}F\frac{\partial^2 W}{\partial\phi^2}\left(\psi_L^T\sigma^2\eta-\eta^T\sigma^2\psi_L\right)\\
&+\frac{i}{2}\frac{\partial^3 W}{\partial\phi^3}\left(-i\eta^T\sigma^2\psi_L\right)\left(\psi_L^T\sigma^2\psi_L\right)+c.c.\\
=&-i\partial_\mu\left(\eta^\dagger\bar{\sigma}^\mu\psi_L\frac{\partial W}{\partial\phi}\right)+\frac{i}{2}\frac{\partial^3 W}{\partial\phi^3}\left(-i\eta^T\sigma^2\psi_L\right)\left(\psi_L^T\sigma^2\psi_L\right)+c.c.\\
=&-i\partial_\mu\left(\eta^\dagger\bar{\sigma}^\mu\psi_L\frac{\partial W}{\partial\phi}\right)+\frac{1}{2}\frac{\partial^3 W}{\partial\phi^3}\eta^T\sigma^2\psi_L\psi_L^T\sigma^2\psi_L+c.c.\\
=&\partial_\mu\left(-i\eta^\dagger\bar{\sigma}^\mu\psi_L\frac{\partial W}{\partial\phi}+c.c.\right)
\end{aligned}
\tag{41}
$$

其中第二步是依据

$$\psi^T A\eta^*=\left(\psi^T A\eta^*\right)^T=-\eta^\dagger A^T\psi \tag{42}$$

其中第三步是依据

$$\eta^T A\psi=\eta_a A_{ab}\psi_b=-\psi_b\left(A^T\right)_{ba}\eta_a=-\psi^T A^T\eta \tag{43}$$

和

$$\sigma^2\sigma^\mu\sigma^2=\bar{\sigma}^\mu \tag{44}$$

其中最后一步是依据

$$(\psi_L\psi_L^T\sigma^2\psi_L)_a=(\psi_L)_a(-i(\psi_L)_1(\psi_L)_2+i(\psi_L)_2(\psi_L)_1)=-i(\psi_L)_a(\psi_L)_1(\psi_L)_2=0 \tag{45}$$

因 $(\psi_L)_1(\psi_L)_1=(\psi_L)_2(\psi_L)_2=0$

对于 $W=g\phi^3/3$，

$$\mathcal{L}_{int}=gF\phi^2+ig\phi\psi_L^T\sigma^2\psi_L+c.c. \tag{46}$$

通过其场方程 $F+g\left(\phi^*\right)^2=0$ 约去 F，

$$\mathcal{L}_{int}=-g\left(\phi^*\right)^2\phi^2+ig\phi\psi_L^T\sigma^2\psi_L+c.c. \tag{47}$$

$$\mathcal{L}=\partial_\mu\phi^*\partial^\mu\phi-g\left(\phi^*\phi\right)^2+\psi_L^\dagger i\bar{\sigma}\cdot\partial\psi_L+ig\left(\phi\psi_L^T\sigma^2\psi_L-\phi^*\psi_L^\dagger\sigma^2\psi_L^*\right) \tag{48}$$

场方程为

$$
\begin{aligned}
\partial^2\phi+2g\phi^*\phi^2+ig\psi_L^\dagger\sigma^2\psi_L^*=0\\
i\bar{\sigma}\cdot\partial\psi_L-2ig\phi^*\sigma^2\psi_L^*=0
\end{aligned}
\tag{49}
$$

习题 8

设 ψ_R 是一个右手外尔旋量场，证明如下量在旋转变换下是一个空间矢量：

$$\psi_R^\dagger \vec{\sigma} \psi_R \tag{50}$$

解：

由

$$\begin{aligned} & e^{\frac{i}{2}\theta\sigma^j} \sigma^k e^{-\frac{i}{2}\theta\sigma^j} \\ &= \left(\mathbf{1}\cos\frac{\theta}{2} + i\sigma^j \sin\frac{\theta}{2}\right) \ \sigma^k \left(\mathbf{1}\cos\frac{\theta}{2} - i\sigma^j \sin\frac{\theta}{2}\right) \\ &= \begin{cases} \sigma^k & j = k \\ \sigma^k \cos\theta - \epsilon_{jkl}\sigma^l \sin\theta & j \neq k \end{cases} \end{aligned} \tag{51}$$

则

$$\psi_R^\dagger \vec{\sigma} \psi_R \to \psi_R^\dagger e^{\frac{i}{2}\theta\hat{n}\cdot\vec{\sigma}} \vec{\sigma} e^{-\frac{i}{2}\theta\hat{n}\cdot\vec{\sigma}} \psi_R \tag{52}$$

在旋转变换下与空间矢量一致。

习题 9

定义如下的矩阵：

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] \tag{53}$$

验证其满足洛伦兹群的李代数关系：

$$\left[S^{\mu\nu}, S^{\rho\sigma}\right] = i\left(g^{\nu\rho} S^{\mu\sigma} - g^{\mu\rho} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\rho} + g^{\mu\sigma} S^{\nu\rho}\right) \tag{54}$$

解：

注意到 $S^{\mu\nu} = -S^{\nu\mu}$，

$$\begin{aligned} \left[S^{\mu\nu}, S^{\rho\sigma}\right] =& -\frac{1}{16}\left[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu, \gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho\right] \\ =& -\frac{1}{16}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho + \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho \right. \\ & \left. -\gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\sigma\gamma^\nu\gamma^\mu + \gamma^\sigma\gamma^\rho\gamma^\mu\gamma^\nu - \gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu\right) \end{aligned} \tag{55}$$

当 $\mu \neq \nu \neq \rho \neq \sigma$ 时，$\left[S^{\mu\nu}, S^{\rho\sigma}\right] = 0$;
当 $\mu = \rho \neq \nu \neq \sigma$ 时，$\left[S^{\mu\nu}, S^{\rho\sigma}\right] = \frac{1}{4}\{\gamma^\mu, \gamma^\rho\}\gamma^\nu\gamma^\sigma = -ig^{\mu\rho}S^{\nu\sigma}$;
当 $\mu = \nu$ 或 $\rho = \sigma$ 时，$\left[S^{\mu\nu}, S^{\rho\sigma}\right] = 0$;
那么

$$\left[S^{\mu\nu}, S^{\rho\sigma}\right] = i\left(g^{\nu\rho} S^{\mu\sigma} - g^{\mu\rho} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\rho} + g^{\mu\sigma} S^{\nu\rho}\right) \tag{56}$$