第七次作业参考答案

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2024 年 1 月 7 日

习题 1

光学定理联系了朝前散射振幅与总散射截面。以 $2 \to 2$ 朝前散射为例，有如下关系：

$$\operatorname{Im} M\left(p_1 p_2 \to p_1 p_2\right)=2 E_{\mathrm{cm}} p_{\mathrm{cm}} \sigma_{\mathrm{tot}}\left(p_1 p_2 \to \text{ anything }\right) \tag{1}$$

其中 $M\left(p_1 p_2 \to p_1 p_2\right)$ 是 $p_1+p_2 \to p_1+p_2$ 的散射振幅，E_{cm} 是质心系能量，p_{cm} 是质心系动量，σ_{tot} $\left(p_1 p_2 \to \text{anything}\right)$ 是总散射截面。请以 $\lambda \phi^4$ 理论为例，

$$\mathcal{L}=\frac{1}{2} \partial_\mu \phi \partial^\mu \phi-\frac{1}{2} m^2 \phi^2-\frac{\lambda}{4!} \phi^4 \tag{2}$$

在 $\mathcal{O}\left(\lambda^2\right)$ 阶水平通过实际计算验证上述关系。

解：

首先计算散射截面，在 $\mathcal{O}\left(\lambda^2\right)$ 阶，只需要计算一阶费曼图，

$$\sigma\left(p_1 p_2 \to \text{anything}\right)=\frac{\lambda^2}{32 \pi E_{cm}^2} \tag{3}$$

因此右边为 $\frac{\lambda^2 p_{cm}}{16 \pi E_{cm}}$.

接下来，计算左边，不难推断仅 s 道有贡献

$$\begin{aligned}
i \mathcal{M}_s &=\frac{1}{2}(-i \lambda)^2 \int \frac{d^4 k}{(2 \pi)^4} \frac{i}{k^2-m^2+i \varepsilon} \frac{i}{\left(p_1+p_2-k\right)^2-m^2+i \varepsilon} \\
&=\frac{\lambda^2}{2} \int \frac{d^4 l}{(2 \pi)^4} \int d x \frac{1}{\left(l^2-\Delta+i \varepsilon\right)^2} \\
&\rightarrow \frac{\lambda^2}{2} \int d x \int \frac{d^d l}{(2 \pi)^d} \frac{1}{\left(l^2-\Delta+i \varepsilon\right)^2} \\
&=\frac{\lambda^2}{2} \int d x \frac{i}{(4 \pi)^{d / 2}} \Gamma(2-d / 2)\left(\frac{1}{\Delta}\right)^{2-d / 2} \\
&=\frac{\lambda^2}{2} \int d x \frac{i}{(4 \pi)^2} \Gamma(\epsilon)\left(\frac{1}{\Delta}\right)^\epsilon \\
&=\frac{\lambda^2}{2} \int d x \frac{i}{(4 \pi)^2}\left(\frac{1}{\epsilon}-\gamma+\mathcal{O}(\epsilon)\right)(1-\epsilon \log \Delta+\cdots)
\end{aligned} \tag{4}$$

其中约定 $\Delta = x(x-1)s + m^2$, $s=(p_1+p_2)^2$, $d = 4-2\epsilon$. 那么,

$$\begin{aligned}
\operatorname{Im} M\left(p_1 p_2 \rightarrow p_1 p_2\right) &= -\frac{\lambda^2}{2(4\pi)^2}\int_0^1 dx \operatorname{Im}\left[\log((x(x-1)s+m^2))\right] \\
&= -\frac{\lambda^2}{2(4\pi)^2}\int_{1/2-\sqrt{1/4-m^2/s}}^{1/2+\sqrt{1/4-m^2/s}} dx(-\pi) \\
&= \frac{\lambda^2}{32\pi}\sqrt{1-4m^2/s} \\
&= \frac{\lambda^2 p_{cm}}{16\pi E_{cm}}
\end{aligned} \tag{5}$$

习题 2

在维数正规化中计算 $\mathcal{O}\left(e^2\right)$ 阶的电子（质量为 m ）与类空虚光子 (无质量) 高能散射的概率。虚光子的四动量为 q^μ，满足 $q^2<0$ 且 $-q^2 \rightarrow \infty$。分别考虑末态只观测到电子的概率，和末态观测到电子与光子的概率

$$\frac{d\sigma}{d\Omega}\left[e(p)+\gamma^*(q) \rightarrow e'\left(p'\right)\right] \tag{6}$$

$$\frac{d\sigma}{d\Omega}\left[e(p)+\gamma^*(q) \rightarrow e'\left(p'\right)+\gamma(k)\right] \tag{7}$$

且末态光子仅包含能量小于一定阈值的光子，$k^0<\omega$ 。其中 Ω 是电子的散射角。验证红外发散在两者之和中相消。计算结果仅保留发散项。

解:

在首阶贡献中

$$i\mathcal{M}_0 = -ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(q) \tag{8}$$

记对应的微分散射截面为 $\left(\frac{d\sigma}{d\Omega}\right)_0$, 不含发散。仅保留发散部分，其一阶修正为

$$\begin{aligned}
i\mathcal{M}_1 &\equiv \bar{u}(p')\delta\Gamma^\mu u(p)\tilde{A}_\mu(q) \\
&= \int \frac{d^dk}{(2\pi)^d}\frac{-ig_{\nu\rho}}{(k-p)^2-\mu^2+i\varepsilon}\bar{u}(p')(-ie\gamma^\nu)\frac{i(\not{k}'+m)}{k'^2-m^2+i\varepsilon}(-ie\gamma^\mu)\frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}(-ie\gamma^\rho)u(p)\tilde{A}_\mu(q) \\
&= -ie\cdot 2ie^2\int\frac{d^dl}{(2\pi)^d}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3}\bar{u}(p')\gamma^\mu(-\frac{d-2}{d}l^2 \\
&\quad +(1-x)(1-y)q^2+(1-4z+z^2)m^2)u(p)\tilde{A}_\mu(q)+\cdots \\
&= i\mathcal{M}_0 2ie^2\int\frac{d^dl}{(2\pi)^d}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3}(-\frac{d-2}{d}l^2 \\
&\quad +(1-x)(1-y)q^2+(1-4z+z^2)m^2) \\
&= i\mathcal{M}_0 2ie^2\int_0^1 dxdydz\delta(x+y+z-1)(\frac{2-d}{2}\frac{i}{(4\pi)^{d/2}}\Gamma(2-d/2)\left(\frac{1}{\Delta}\right)^{2-d/2} \\
&\quad +((1-x)(1-y)q^2+(1-4z+z^2)m^2)\frac{-i}{(4\pi)^{d/2}}\Gamma(3-d/2)\left(\frac{1}{\Delta}\right)^{3-d/2})
\end{aligned} \tag{9}$$

其中，$D=l^2-\Delta+i\epsilon$, $\Delta=-xyq^2+(1-z)^2m^2+\mu^2 z$. 记 $4-2\epsilon=d$, 结果仅保留红外区域对数发散项

$$
\begin{aligned}
i\mathcal{M}_1 = & i\mathcal{M}_0 2ie^2\frac{-i}{(4\pi)^{d/2}}\int_0^1 dxdydz\delta(x+y+z-1)\left(\frac{d-2}{2}(\frac{1}{\epsilon}-\gamma-\log\Delta)\right.\\
&\left.+((1-x)(1-y)q^2+(1-4z+z^2)m^2)\Gamma(1+\epsilon)\left(\frac{1}{\Delta}\right)^{1+\epsilon}\right)\\
&\approx i\mathcal{M}_0 2ie^2\frac{-i}{(4\pi)^2}\int_0^1 dxdydz\delta(x+y+z-1)\left(\frac{(1-x)(1-y)q^2+(1-4z+z^2)m^2}{\Delta^{1+\epsilon}}\right)\\
&\approx i\mathcal{M}_0 2ie^2\frac{-i}{(4\pi)^2}\int_0^1 dydz\left(\frac{(y+z)(1-y)q^2+(1-4z+z^2)m^2}{(-(1-y-z)yq^2+(1-z)^2m^2+\mu^2 z)^{1+\epsilon}}\right)\\
&\xrightarrow[y=\mu/M\xi\sqrt{x}]{z=1-\mu/M\sqrt{x}} i\mathcal{M}_0 2ie^2\frac{-i}{(4\pi)^2}\left(\mu^2/M^2\right)^{-\epsilon}\int_0^1\frac{1}{2}dxd\xi\frac{q^2-2m^2}{(1+(m^2-(1-\xi)\xi q^2)x)^{1+\epsilon}}\\
&= i\mathcal{M}_0\frac{\alpha}{2\pi}\left(-\frac{\left(\mu^2/M^2\right)^{-\epsilon}}{\epsilon}\right)\int_0^1 d\xi\frac{q^2/2-m^2}{m^2-(1-\xi)\xi q^2}\\
&= i\mathcal{M}_0\frac{\alpha}{2\pi}\left(-\frac{1}{\epsilon}-\log\frac{\mu^2}{M^2}\right)\int_0^1 d\xi\frac{q^2/2-m^2}{m^2-(1-\xi)\xi q^2}\\
&\approx -i\mathcal{M}_0\frac{\alpha}{2\pi}\int_0^1 d\xi\frac{m^2-q^2/2}{m^2-q^2(1-\xi)\xi}\log\frac{M^2}{\mu^2}\\
&\xrightarrow{-q^2\to\infty} -i\mathcal{M}_0\frac{\alpha}{4\pi}\log\left(\frac{-q^2}{m^2}\right)\log\frac{M^2}{\mu^2}
\end{aligned}
\tag{10}
$$

其中,$M^2=-q^2$或m^2. 当然, 更方便的可以在红外积分区域展开圈积分,

$$
\begin{aligned}
i\mathcal{M}_1 &\equiv \bar{u}(p')\delta\Gamma^\mu u(p)\tilde{A}_\mu(q)\\
&= -ie2ie^2\int\frac{d^dk}{(2\pi)^d}\frac{\bar{u}(p')\left[\not{k}\gamma^\mu\not{k}'+m^2\gamma^\mu-2m(k+k')^\mu\right]u(p)}{((k-p)^2+i\varepsilon)(k'^2-m^2+i\varepsilon)(k^2-m^2+i\varepsilon)}\tilde{A}_\mu(x)\\
&\xrightarrow[l=k-p]{l^2\approx\mu^2\to 0} -ie2ie^2\int\frac{d^dl}{(2\pi)^d}\frac{\bar{u}(p')\gamma^\mu u(p)(p\cdot p')}{(l^2+i\varepsilon)(l\cdot p'+i\varepsilon)(l\cdot p+i\varepsilon)}\tilde{A}_\mu(x)\\
&= i\mathcal{M}_0 2ie^2\int\frac{d^dl}{(2\pi)^d}\frac{(p\cdot p')}{(l^2+i\epsilon)(l\cdot p'+i\varepsilon)(l\cdot p+i\varepsilon)}\\
&\overset{l=u\mu}{=} i\mathcal{M}_0 2ie^2\mu^{d-4}\int\frac{d^du}{(2\pi)^d}\frac{m^2-q^2/2}{(u^2+i\epsilon)(u\cdot(p+q)+i\varepsilon)(u\cdot p+i\varepsilon)}\\
&\overset{-q^2\to\infty}{=} -i\mathcal{M}_0\frac{\alpha}{4\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{M^2}{\mu^2}\right)
\end{aligned}
\tag{11}
$$

其中，计算最后一步时候选择一个参考系，使得 $p^0=p'^0=E$，则动量可以写作

$$
u^\mu=(u^0,\vec{u}),\quad p^\mu=E(1,\hat{v}),\quad p'^\mu=E(1,\hat{v}')
\tag{12}
$$

则积分为

$$\begin{aligned}
&\int \frac{d^d u}{(2\pi)^d} \frac{(p\cdot p)}{(u^2)(u\cdot p')(u\cdot p)} \\
=&\int \frac{d^{d-1}\vec{u}}{(2\pi)^d} \int du^0 \frac{1}{(u^0)^2-(\vec{u})^2} \frac{1-\hat{v}\cdot\hat{v}'}{(u^0-\vec{u}\cdot\hat{v})(u^0-\vec{u}\cdot\hat{v}')} \\
=&2\pi i \int \frac{d^{d-1}\vec{u}}{(2\pi)^d} \frac{1}{2(\vec{u})^3} \frac{1-\hat{v}\cdot\hat{v}'}{(1-\hat{u}\cdot\hat{v})(1-\hat{u}\cdot\hat{v}')} \\
=&2\pi i \int \frac{|\vec{u}|^{d-2} d|\vec{u}|}{(2\pi)^d} \frac{1}{2|\vec{u}|^3} \int d\Omega_{\hat{u}} \frac{1-\hat{v}\cdot\hat{v}'}{(u^0-\hat{u}\cdot\hat{v})(u^0-\hat{u}\cdot\hat{v}')} \\
\overset{x=|\vec{u}|^2}{=}&2\pi i \frac{1}{(2\pi)^{4-2\epsilon}} \int \frac{x^{-\epsilon}dx}{4x} \int d\Omega_{\hat{u}} \frac{1-\hat{v}\cdot\hat{v}'}{(u^0-\hat{u}\cdot\hat{v})(u^0-\hat{u}\cdot\hat{v}')} \\
=&\frac{i}{16\pi^2\epsilon} \log\left(\frac{-q^2}{m^2}\right)
\end{aligned} \tag{13}$$

类似的积分在计算末态出射软光子的时候还会出现。综上，

$$\frac{d\sigma}{d\Omega}\left[e(p)+\gamma^*(q)\to e'(p')\right] = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(1-\frac{\alpha}{\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{\mu^2}\right)\right) \tag{14}$$

下面计算, 末态观测到软光子的微分散射截面, 对应的散射振幅为

$$\begin{aligned}
i\mathcal{M} &= -ie\bar{u}(p')\left(-ie\gamma^\mu \frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2}\gamma^\nu\epsilon_\nu^*(k) + \gamma^\nu\epsilon_\nu^* \frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}(-ie\gamma^\mu)\right)u(p)\tilde{A}_\mu(q) \\
&= -ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(q)\cdot\left[e\left(\frac{p'\cdot\epsilon^*}{p'\cdot k}-\frac{p\cdot\epsilon^*}{p\cdot k}\right)\right]
\end{aligned} \tag{15}$$

$$d\sigma\left[e(p)+\gamma^*(q)\to e'(p')+\gamma(k)\right] = d\sigma_0 \int \frac{d^3k}{(2\pi)^3 2k} \sum_\lambda e^2 \left|\vec{\epsilon}_\lambda\cdot\left(\frac{\vec{p}'}{p'\cdot k}-\frac{\vec{p}}{p\cdot k}\right)\right| \tag{16}$$

再次遇到了相似的积分，则可得到著名的结果

$$\frac{d\sigma}{d\Omega}\left[e(p)+\gamma^*(q)\to e'(p')+\gamma(k)\right] \overset{-q^2\to\infty}{\approx} \left(\frac{d\sigma}{d\Omega}\right)_0 \frac{\alpha}{\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{\mu^2}\right) \tag{17}$$

容易发现公式 (14) 和 (17) 红外发散相消.