Constructing Canonical Feynman Integrals with Intersection Theory

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- Background
- Intersection theory
- Examples
- Conclusion

motivation

- More and more precise experiment in LHC and HL-LHC or other accelerator may be build in the future call for more precise theoretical prediction, more efficient method to calculate amplitude.

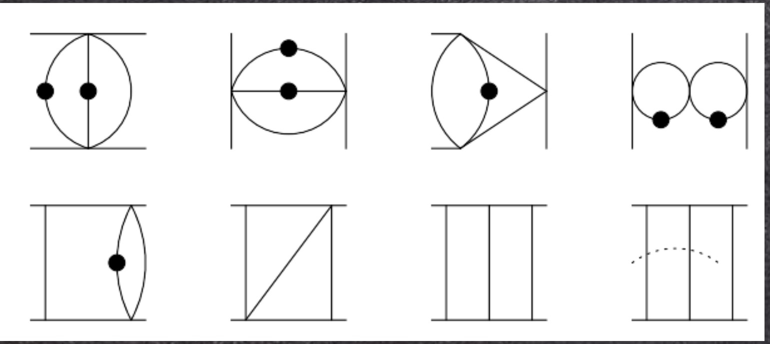
- Better understanding of Feynman integral’s mathematic structure will also benefit fundamental understanding of quantum field theory.

IBP (integrate by part) method find many relations between Feynman integrals which greatly reduce the number of Feynman integrals people need to calculate.

IBP combine with differential equations transform the problem to find the solutions of the differential equations of master integrals

$$\frac{d}{ds_i}\vec{f} = A\vec{f}$$

canonical-form of differential equations make them easier to solve to any order in eps



$$\partial_x f = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] f$$

$$
\begin{aligned}
f_1 &= -\epsilon^2 (-s)^{2\epsilon}\, t\, I_{0,2,0,0,0,0,0,1,2}\,,\\
f_2 &= \epsilon^2 (-s)^{1+2\epsilon}\, I_{0,0,2,0,1,0,0,0,2}\,,\\
f_3 &= \epsilon^3 (-s)^{1+2\epsilon}\, I_{0,1,0,0,1,0,1,0,2}\,,\\
f_4 &= -\epsilon^2 (-s)^{2+2\epsilon}\, I_{2,0,1,0,2,0,1,0,0}\,,\\
f_5 &= \epsilon^3 (-s)^{1+2\epsilon}\, t\, I_{1,1,1,0,0,0,0,1,2}\,,\\
f_6 &= -\epsilon^4 (-s)^{2\epsilon} (s+t)\, I_{0,1,1,0,1,0,0,1,1}\,,\\
f_7 &= -\epsilon^4 (-s)^{2+2\epsilon}\, t\, I_{1,1,1,0,1,0,1,1,1}\,,\\
f_8 &= -\epsilon^4 (-s)^{2+2\epsilon}\, I_{1,1,1,0,1,-1,1,1,1}\,.
\end{aligned}
$$

$$
a = \begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{3}{2} & 0 & 0 & 0 & -2 & 0 & 0 & 0\\
-\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -2 & 0 & 0\\
-3 & -3 & 0 & 0 & 4 & 12 & -2 & 0\\
\frac{9}{2} & 3 & -3 & -1 & -4 & -18 & 1 & 1
\end{pmatrix}
$$

$$
b = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\frac{3}{2} & 0 & 3 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0\\
3 & 6 & 6 & 2 & -4 & -12 & 2 & 2\\
-\frac{9}{2} & -3 & 3 & -1 & 4 & 18 & -1 & -1
\end{pmatrix}
$$

Johannes M. Henn, 1304.1806

Uniform transcendentality (UT) and dlog

Solution of canonical form differential equations usually are MPL

$$G(a_1; z) = \int_0^z \frac{dt}{t - a_1}, \qquad a_1 \neq 0.$$

$$G(a_1, \ldots a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t),$$

$G(a_1, a_2, \ldots, a_n; z)$ transcendental weight n .

Canonical form give the solution of canonical basis in this form:

$$1 + \varepsilon g_1 + \varepsilon^2 g_2 + \ldots \qquad g_i \sim weight\ i$$

define ε's weight is -1, Master integrals are UT

$$\mathcal{T}(f_1 f_2) = \mathcal{T}(f_1)\mathcal{T}(f_2)$$

$$\pi \sim log(-1) \quad weght\ \ 1$$

Claude Duhr, 1411.7538

Uniform transcendentality (UT) and dlog

Choose dlog-form Feynman integrals as Master integrals make differential equations automatically canonical form

$$d\log\text{-form} \quad : \quad d\log(f_1) \wedge d\log(f_2) \wedge \cdots \wedge d\log(f_n)$$

Strategy: variable by variable

For the constructed i variables, we hope it takes the form:

$$\varphi^{(i)} = g(\boldsymbol{z})\sqrt{\Omega_i(\boldsymbol{z}_{j,j>i})} \prod_{k,k\leq i} dz_k$$

So that we may construct next suitable variable's dlog as

$$\varphi^{(i+1)} = \varphi^{(i)} \frac{g_{i+1}(\boldsymbol{z})}{\sqrt{\Omega_i(\boldsymbol{z}_{j,j>i})}} \sqrt{\Omega_{i+1}(\boldsymbol{z}_{j,j>i+1})} dz_{i+1}$$

Baikov representation

integration variable transformation $$\prod_{j=1}^{L} d^d l_j \;\to\; \prod_{i}^{n} dD_i$$

$$\pi^{d/2} I(N,\{\alpha_i\},d) = \int N \prod_{i=0}^{E} \frac{1}{D_i^{\alpha_i}} d^d l \;=\; \int_\sigma \frac{N\Omega_{d-E}}{2^{E+1}} \prod_{i=1}^{E+1} \frac{dD_i}{D_i^{\alpha_i}} \frac{|F|^{(d-E-2)/2}}{|U|^{(d-E-1)/2}}$$

$$F = (-1)^E det \begin{pmatrix} l\cdot l & l\cdot p_1 & l\cdot p_2 & \cdots & l\cdot p_E \\ l\cdot p_1 & p_1\cdot p_1 & p_1\cdot p_2 & \cdots & p_1\cdot p_E \\ \vdots & & & & \vdots \\ l\cdot p_E & & \cdots & & p_E\cdot p_E \end{pmatrix} \quad U = (-1)^{E-1} det \begin{pmatrix} p_1\cdot p_1 & p_1\cdot p_2 & \cdots & p_1\cdot p_E \\ p_1\cdot p_2 & & & \vdots \\ \vdots & & & \vdots \\ p_1\cdot p_E & \cdots & & p_E\cdot p_E \end{pmatrix}$$

Cut: $$\int dD_i \;\to\; \oint_{D_i=0} dD_i$$

H. Frellesvig and C. G. Papadopoulos, 1701.07356

Loop by loop Baikov representation

$$I \sim \int \cdots \int \prod_{j=1}^{E_2+1} \frac{dD_{2j}}{D_{2j}^{\alpha_{2j}}} \frac{|F_2|^{(d-E_2-2)/2}}{|U_2|^{(d-E_2-1)/2}} \int \prod_{i=1}^{E_1+1} \frac{dD_{1i}}{D_{1i}^{\alpha_{1i}}} \frac{|F_1|^{(d-E_1-2)/2}}{|U_1|^{(d-E_1-1)/2}}$$

More than one master integrals in one sector call for different denominator besides D_i

$$I \sim \int \cdots \int \prod_{j=1}^{E_2+1} \frac{dD_{2j}}{D_{2j}^{\alpha_{2j}}} \frac{|F_2|^{(d-E_2-2)/2}}{|U_2|^{(d-E_2-1)/2}} \int \prod_{i=1}^{E_1+1} \frac{dD_{1i}}{D_{1i}^{\alpha_{1i}}} \frac{|F_1|^{(d-E_1-2)/2}}{|U_1|^{(d-E_1-1)/2}} \frac{1}{\mathrm{G_i}}$$

Is this form a linear combination of Feynman integral?

If the answer is Yes,

How can we transform it to Feynman integrals?

Intersection theory

Why need Intersection theory?

In mutli-loop cases' dlog construction we encounter some expression that is not explicitly a Feynman intergral.
Intersection theory extend IBP to these cases and help us transform them back to Feynman intergral, which just like do a projection in a linear space.
And it tell us the number of master integrals

P. Mastrolia and S. Mizera, 1810.03818
H. Frellesvig and et al., 1901.11510
H. Frellesvig and et al., 1907.02000
S. Mizera, 1910.11852
S. Weinzierl, 2002.01930
S. Mizera 2002.10476
H. Frellesvig and et al., 2008.04823

hypergeometric functions

$$I = \int_{\mathcal{C}} u(\mathbf{z})\, \varphi(\mathbf{z})$$

Boundary C located at where pi=0

$$u = \prod_i p_i^{\nu_i} \quad \nu_i \in \mathbb{C}$$

$$\varphi = \frac{q}{\prod_i p_i^{n_i}} \bigwedge_j dz_j \quad n_i \in \mathbb{N}_0$$

Baikov representation

$$\int_\sigma \frac{N\Omega_{d-E}}{2^{E+1}} \prod_{i=1}^{E+1} \frac{dD_i}{D_i^{\alpha_i}} \frac{|F|^{(d-E-2)/2}}{|U|^{(d-E-1)/2}}$$

equivalence classes defined by IBP

$$0 = \int_{\mathcal{C}} d\,(u\,\xi) = \int_{\mathcal{C}} (du \wedge \xi + u\,d\xi) = \int_{\mathcal{C}} u\left(\frac{du}{u}\wedge + d\right)\xi \equiv \int_{\mathcal{C}} u\,\nabla_\omega \xi.$$

$$\langle\varphi| : \varphi \sim \varphi + \nabla_w \xi$$

$$\nabla_w \equiv d + w\wedge$$

$$\omega = d\ln u = \sum_{j=1}^{n} \omega_j dz_j,$$

$\langle\varphi|$ form a linear space H^n_ω whose basis $\langle e_i|$ is corresponding to MIs.

Projection

A set of complete master integrals form a basis of H_ω^n

Any element belong to H_ω^n can be decomposed to this basis

$$\langle \varphi | = \sum_{i=1}^{\nu} c_i \langle e_i |$$

$$c_i = \sum_{j=1}^{\nu} \langle \varphi | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji}$$

$$\mathbf{C}_{ij} = \langle e_i | h_j \rangle$$

The definition of Inner product play the core role !!!

Inner product（intersection number）

n-form：

$$\langle \varphi_L | \varphi_R \rangle = \left(\frac{1}{2\pi i} \right)^n \int_{\mathcal{C}} \varphi_L^c \wedge \varphi_R$$

1-form：

$$\varphi_L^c = \varphi_L - \nabla_\omega \sum_{p \in \mathcal{P}} \Theta(|z-p|^2 - \varepsilon^2) \nabla_\omega^{-1} \varphi_L$$

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathcal{P}} \mathrm{Res}_{\mathrm{z=p}} \, (\nabla_\omega^{-1} \varphi_{\mathrm{L}}) \varphi_{\mathrm{R}}$$

$$\mathcal{P} = \{ \ poles \ of \ \omega \ \}$$

n-form intersection number is also given in reference

Inner product（intersection number）

1-form：

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathcal{P}} \mathrm{Res_{z=p}}\ (\nabla_\omega^{-1} \varphi_\mathrm{L}) \varphi_\mathrm{R}$$

e.g.

$$\langle \varphi_1 | \varphi_2 \rangle = \langle \varphi_1 + \nabla_\omega \xi | \varphi_2 \rangle$$

$$u = \prod_i (z - c_i)^{\alpha_i}$$

$$\varphi_L = \nabla_\omega \xi$$

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathcal{P}} \mathrm{Res_{z=p}}\ \xi \varphi_\mathrm{R}$$

Notice that $\xi \varphi_R \sim \frac{q}{\prod_i (z - c_i)^{n_i}} dz$ The result is 0 !

intersection number

$$\langle \varphi_L | \varphi_R \rangle_\omega = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p} \Big(\psi_p \, \varphi_R \Big)$$

$$\nabla_{\omega_p} \psi_p = \varphi_{L,p}$$

defining $\tau \equiv z - p$, and the *ansatz*,

$$\psi_p = \sum_{j=\min}^{\max} \psi_p^{(j)} \tau^j + \mathcal{O}\left(\tau^{\max+1}\right) \ ,$$

$$\min = \mathrm{ord}_p(\varphi_L) + 1 \ , \qquad \max = -\mathrm{ord}_p(\varphi_R) - 1$$

Number of Master Integrals

$$\omega_i = \frac{d}{dz_i} log(u(\boldsymbol{z}))$$

Number of Master Integrals is the number of solution of

$$\omega_i = 0$$

Roman N. Lee, Andrei A. Pomeransky 1308.6676

examples

$$I = \int_{\mathcal{C}} \prod_i \frac{dD_i}{D_i^{\alpha_i}} \prod_j G_j^{\gamma_j + \beta_j \varepsilon}$$

$$\gamma_j \in \mathbb{Z}/2 \qquad \alpha_i, \beta_j \in \mathbb{Z}$$

1-form

$$u(z) = \frac{\mathcal{K}_1^{\epsilon}}{\mathcal{K}_0} \prod_{j=0}^{\nu} (z - c_j)^{-\gamma_j' - \beta_j' \epsilon}$$

canonical basis
in all 1-form cases
(without elliptic
integral)

$$\frac{\partial}{\partial x}\log\frac{1+\sqrt{\frac{(x_2-c)(x_1-x)}{(x_1-c)(x_2-x)}}}{1-\sqrt{\frac{(x_2-c)(x_1-x)}{(x_1-c)(x_2-x)}}}=\frac{\sqrt{(x_1-c)(x_2-c)}}{(x-c)\sqrt{(x-x_1)(x-x_2)}},$$

$$\frac{\partial}{\partial x}\log\frac{1+\sqrt{\frac{(x_1-x)}{(x_2-x)}}}{1-\sqrt{\frac{(x_1-x)}{(x_2-x)}}}=\frac{1}{\sqrt{(x-x_1)(x-x_2)}},$$

$$u=\prod_i^n(z-c_i)^{\beta_i\varepsilon}$$

$$e_i=\frac{1}{z-c_i}\qquad i=1,2,\cdots,n-1$$

$$u=\prod_i^{n-1}(z-c_i)^{\beta_i\varepsilon}\ *\ (z-c_n)^{-\frac{1}{2}+\beta_n\varepsilon}$$

$$e_i=\frac{\sqrt{c_i-c_n}}{z-c_i}\qquad i=1,2,\cdots,n-1$$

$$u=\prod_i^{n-2}(z-c_i)^{\beta_i\varepsilon}\ *\ (z-c_{n-1})^{-\frac{1}{2}+\beta_{n-1}\varepsilon}(z-c_n)^{-\frac{1}{2}+\beta_n\varepsilon}$$

$$e_i=\frac{\sqrt{c_i-c_{n-1}}\sqrt{c_i-c_n}}{z-c_i}\qquad i=1,2,\cdots,n-2$$

$$e_{n-1}=1$$

Canonical basis for one loop

$$F_{a_1,\ldots,a_{E+1}} = \frac{1}{(4\pi)^{E/2}\Gamma((d-E)/2)} \int \frac{[G(\boldsymbol{z})]^{(d-E-2)/2}}{\mathcal{K}^{(d-E-1)/2}} \prod_{i=1}^{E+1} \frac{dz_i}{z_i^{a_i}},$$

$$u(\boldsymbol{z}) = [G(\boldsymbol{z})]^{(2-E)/2-\epsilon}\, \mathcal{K}^{(E-3)/2+\epsilon}$$

$$u(\boldsymbol{z}) = [G(\boldsymbol{z})]^{(2-E)/2-\epsilon}\,\mathcal{K}^{(E-3)/2+\epsilon}$$

E is even：

$$\varphi(\boldsymbol{z}) = \mathcal{K}^{(3-E)/2}\,[G(\boldsymbol{z})]^{(E-2)/2}\bigwedge_{i=1}^{E+1} d\log(z_i)$$

E is odd：

$$\varphi(\boldsymbol{z}) = \sqrt{G(\boldsymbol{0})}\,\mathcal{K}^{(3-E)/2}\,[G(\boldsymbol{z})]^{(E-3)/2}\bigwedge_{i=1}^{E+1}\frac{dz_i}{z_i}$$

$$u(\boldsymbol{z})\varphi(\boldsymbol{z}) = \mathcal{K}^{\epsilon}\sqrt{G(\boldsymbol{0})}\bigwedge_{i=1}^{E+1}\frac{dz_i}{z_i}[G(\boldsymbol{z})]^{-1/2-\epsilon}$$

$$= \left[\frac{G(\boldsymbol{z})}{\mathcal{K}}\right]^{-\epsilon}\bigwedge_{i=1}^{E+1}\frac{dz_i}{z_i}\sqrt{\frac{G(\vec{0}_i, z_{i+1}, \ldots, z_{E+1})}{G(\vec{0}_{i-1}, z_i, \ldots, z_{E+1})}}$$

$$\frac{\sqrt{x_1 x_2}}{x\sqrt{(x_1-x)(x_2-x)}}$$

Examples: massless double box

$$\{k_1^2,\,(k_1+p_1)^2,\,(k_1+p_1+p_2)^2,\,(k_1+k_2)^2,\,k_2^2,\,(k_2-p_3)^2,\,(k_2-p_1-p_2)^2,\,z=(k_2-p_1)^2,\,D_9\}$$

$$u(z)=\frac{1}{s^2}\left(\frac{t(s+t)}{s^2}\right)^{\epsilon}z^{-1-\epsilon}(s+z)^{\epsilon}(t-z)^{-1-2\epsilon}$$

$$\phi_1=s^2 z dz\,,\quad \phi_2=s^2(t-z)dz\,.$$

Examples: massless double box

$$E_1 = F_{1,1,1,1,1,1,1,0,0} \text{ and } E_2 = F_{1,2,1,1,1,1,1,0,0}.$$

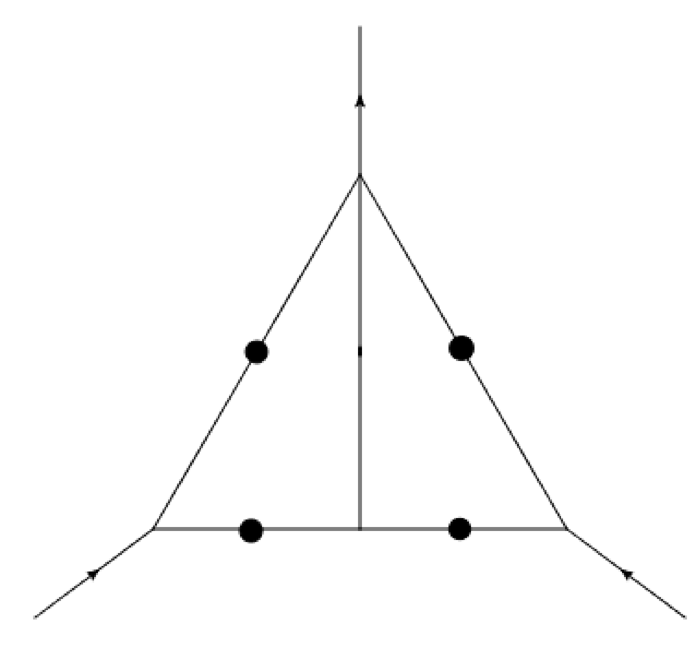
$$I_1 = -\frac{s(1+3\epsilon)}{2\epsilon} E_1 + \frac{st(1+\epsilon)}{2\epsilon(1+2\epsilon)} E_2 \,,$$
$$I_2 = \frac{s(1+3\epsilon)+2\epsilon t}{2\epsilon} E_1 - \frac{st(1+\epsilon)}{2\epsilon(1+2\epsilon)} E_2 \,.$$

$$\frac{\partial}{\partial s}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \epsilon \begin{pmatrix} -\frac{2}{s} & \frac{1}{s+t} \\ \frac{2}{s} & -\frac{s+2t}{s(s+t)} \end{pmatrix}, \quad \frac{\partial}{\partial t}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \epsilon \begin{pmatrix} 0 & -\frac{s}{t(s+t)} \\ -\frac{2}{t} & -\frac{s}{t(s+t)} \end{pmatrix}$$

Examples: 4 mass triangle

$$\{k_1^2 - m^2,\ (k_1 - k_2)^2,\ (k_1 + p_2)^2 - m^2,\ (k_2 - p_1)^2 - m^2,$$
$$(k_2 + p_2)^2 - m^2,\ z \equiv k_2^2 - m^2,\ (k_1 - p_1)^2 - m^2\}$$

$$p_1^2 = m_1^2$$
$$p_2^2 = m_2^2$$
$$(p_1 + p_2)^2 = s$$



consider sector of D_i i=1,2,3,4,5

$$u = \frac{F_1^{-\varepsilon}}{U_1^{\frac{1}{2}-\varepsilon}} \frac{F_2^{-\varepsilon}}{U_2^{\frac{1}{2}-\varepsilon}} D_1^{\delta_1} D_2^{\delta_2} D_3^{\delta_3} D_4^{\delta_4} D_5^{\delta_5}$$

$$\frac{dD_i}{D_i^{\alpha_i}} \frac{|F|^{(d-E-2)/2}}{|U|^{(d-E-1)/2}}$$

Examples: 4 mass triangle maximal cut

$$u(z) = \frac{1}{\sqrt{\lambda}}\left(\frac{\lambda}{sm^2}\right)^{\epsilon} z^{-2\epsilon}\left[(z-c_0)(z-c_1)\right]^{-1/2+\epsilon}\left[(z-c_2)(z-c_3)\right]^{-\epsilon}$$

$$c_{0,1} = m_2(m_2 \pm 2m)$$

$$c_{2,3} = \frac{m_1^2 s + m_2^2 s - s^2 \pm \sqrt{s(s-4m^2)\lambda}}{2s}$$

$$\lambda \equiv \lambda(s, m_1^2, m_2^2)$$

$$\hat{\phi}_1(z) = \sqrt{\lambda}, \quad \hat{\phi}_4(z) = \sqrt{\lambda}\frac{\sqrt{c_0 c_1}}{z},$$

$$\hat{\phi}_{2,3}(z) = \sqrt{\lambda}\frac{\sqrt{(c_0 - c_{2,3})(c_1 - c_{2,3})}}{z - c_{2,3}}.$$

Examples: 4 mass triangle maximal cut

$$F_{1,1,1,1,1,0,0},\ F_{2,1,1,1,1,0,0},\ F_{1,2,1,1,1,0,0} \text{ and } F_{1,1,1,2,1,0,0}.$$

$$\hat{e}_1(z) = 1\,,\ \hat{e}_2(z) = \frac{2\epsilon}{z}\,,\ \hat{e}_3(z) = \frac{\epsilon\, m_2^2(z+4m^2-m_2^2)}{m^2 z^2}\,, \hat{e}_4(z) = \epsilon\frac{m_2^2(s+m_1^2-m_2^2)+z(s-m_1^2+m_2^2)}{s(z-c_2)(z-c_3)}\,.$$

Transform to Feynman integrals

$$\begin{aligned}
\langle\phi_1| &= \sqrt{\lambda}\,\langle e_1|\,, \quad \langle\phi_4| = \frac{1}{2\epsilon} m_2\sqrt{m_2^2-4m^2}\sqrt{\lambda}\,\langle e_2|\,,\\
\langle\phi_{2,3}| &= \frac{1}{\epsilon^2\left[s(m_2^2-2m^2)+2m^2(m_1^2-m_2^2)\right]}\\
&\quad\times\Big[(\lambda m^2+m_1^2m_2^2 s)\left[m^2(2\epsilon+1)\,\langle e_3| - 2\epsilon^2\,\langle e_1|\right]\\
&\quad+\frac{\epsilon}{2} s m_2^2(m_2^2-4m^2)(m_1^2+m_2^2-s)\,\langle e_2|\\
&\quad+\frac{\epsilon}{2} s m_2^2(s-4m^2)(m_1^2-m_2^2+s)\,\langle e_4|\Big]\\
&\quad\mp\frac{1}{2\epsilon}\sqrt{\lambda}\sqrt{s(s-4m^2)}\,\langle e_4|\,.
\end{aligned}$$

Examples: 4 mass triangle

$$\rho_1 = \lambda(s, z_4, z_5) - 4sm^2\,,$$
$$\rho_2 = \lambda(m_2^2, z_1, z_3) - 4m_2^2 m^2\,,$$
$$\rho_3 = \lambda(z_2, z_3, z_5) - 4z_2 m^2\,.$$

$$u(\boldsymbol{z}) = \frac{\lambda^{-1/2+\epsilon}\,(z_3+m^2)^{-\epsilon}}{\left[(z-c_0)(z-c_1)\right]^{1/2-\epsilon}} \prod_{i=2}^{5} (z-c_i)^{-\epsilon}$$

$$c_{0,1} = m_2^2 + z_5 \pm 2\sqrt{m_2^2(m^2+z_5)}\,,$$
$$c_{2,3} = \frac{1}{2s}\left[z_4(s-m_1^2+m_2^2) + z_5(s+m_1^2-m_2^2) + s(m_1^2+m_2^2-s) \pm \sqrt{\lambda\,\rho_1}\right],$$
$$c_{4,5} = \frac{1}{2(z_3+m^2)}\left[m_2^2(z_2+z_3-z_5) + z_5(2m^2+z_1+z_3) + (z_1-z_3)(2m^2-z_2+z_3) \pm \sqrt{\rho_2\rho_3}\right]$$

$$\hat{\phi}_2^{(1)}(\boldsymbol{z}) = \sqrt{\lambda}\frac{\sqrt{(c_0-c_2)(c_1-c_2)}}{z-c_2} = \sqrt{\lambda}\frac{\sqrt{\lambda}(s-z_4+z_5) - \sqrt{\rho_1}(s-m_1^2+m_2^2)}{2s(z-c_2)}\,,$$
$$\hat{\phi}_3^{(1)}(\boldsymbol{z}) = \sqrt{\lambda}\frac{\sqrt{(c_0-c_3)(c_1-c_3)}}{z-c_3} = \sqrt{\lambda}\frac{\sqrt{\lambda}(s-z_4+z_5) + \sqrt{\rho_1}(s-m_1^2+m_2^2)}{2s(z-c_3)}\,.$$

n-form dlog construction

$$\hat{\phi}_2^{(1)}(\boldsymbol{z}) = \sqrt{\lambda}\frac{\sqrt{(c_0-c_2)(c_1-c_2)}}{z-c_2} = \sqrt{\lambda}\frac{\sqrt{\lambda}(s-z_4+z_5)-\sqrt{\rho_1}(s-m_1^2+m_2^2)}{2s(z-c_2)},$$

$$\hat{\phi}_3^{(1)}(\boldsymbol{z}) = \sqrt{\lambda}\frac{\sqrt{(c_0-c_3)(c_1-c_3)}}{z-c_3} = \sqrt{\lambda}\frac{\sqrt{\lambda}(s-z_4+z_5)+\sqrt{\rho_1}(s-m_1^2+m_2^2)}{2s(z-c_3)}.$$

we prefer it to have the form:

$$\varphi^{(i)} = g(\boldsymbol{z})\sqrt{\Omega_i(\boldsymbol{z}_{j,j>i})}\prod_{k,k\leq i} dz_k$$

Examples: 4 mass triangle

Linear combination:

$$\hat{\varphi}_2^{(1)}(\boldsymbol{z}) = \hat{\phi}_2^{(1)}(\boldsymbol{z}) + \hat{\phi}_3^{(1)}(\boldsymbol{z}) = s\left[\frac{\partial \log\left((z-c_2)(z-c_3)\right)}{\partial z}\frac{\partial}{\partial z_4} - 2\frac{\partial^2}{\partial z_4 \partial z}\right]\left[(z-c_2)(z-c_3)\right]$$
$$= \frac{\lambda}{2s^2(c_2-z)(z-c_3)}\left[\rho_1(s-m_1^2+m_2^2) + s(c_2+c_3-2z)(s-z_4+z_5)\right],$$

$$\hat{\varphi}_2'(\boldsymbol{z}') = \frac{1}{z_1 z_2 z_3 z_4 z_5}$$

dlog-form: $\varphi_2(\boldsymbol{z}) = \hat{\varphi}_2^{(1)}(\boldsymbol{z})\hat{\varphi}_2'(\boldsymbol{z}')d^6\boldsymbol{z}$

Examples: 4 mass triangle

Linear combination:

$$\hat{\varphi}_3^{(1)}(\boldsymbol{z}) = \hat{\phi}_2^{(1)}(\boldsymbol{z}) - \hat{\phi}_3^{(1)}(\boldsymbol{z}) = \frac{\sqrt{\lambda}\sqrt{\rho_1}}{(z-c_2)(z-c_3)} \frac{\partial}{\partial z_4}\left[(z-c_2)(z-c_3)\right]$$
$$= \frac{\sqrt{\lambda}\sqrt{\rho_1}}{2s^2(z-c_2)(z-c_3)}\left[\lambda(s-z_4+z_5)+s(c_2+c_3-2z)(s-m_1^2+m_2^2)\right]$$

$$\lambda \equiv \lambda(s, m_1^2, m_2^2)$$
$$\rho_1 = \lambda(s, z_4, z_5) - 4sm^2$$

$$\hat{\varphi}_3'(\boldsymbol{z}') = \frac{1}{z_1 z_2 z_3} \frac{\sqrt{s(s-4m^2)}}{z_4 z_5}$$

dlog-form:

$$\varphi_3(\boldsymbol{z}) = \hat{\varphi}_3^{(1)}(\boldsymbol{z}) \frac{\hat{\varphi}_3'(\boldsymbol{z}')}{\sqrt{\rho_1}} d^6 \boldsymbol{z}$$

dlog-form corresponding to c_4, c_5 is similar

Example: sunrise

$$D_1 = l_1^2 - m^2, \quad D_2 = l_2^2 - m^2, \quad D_3 = (l_1 + l_2 + p_1)^2,$$
$$D_4 = (l_2 + p_1)^2 - m^2, \quad D_5 = (l_1 + p_1)^2 - m^2$$

$d = 2 - 2\varepsilon$ consider $G[1, \{1,1,1,0,0\}]$

$$\sim \int dD_5 \frac{dD_1}{D_1} \frac{1}{\sqrt{F_2}} \left(\frac{F_2}{U_2}\right)^{-\varepsilon} \int \frac{dD_2}{D_2} \frac{dD_3}{D_3} \frac{1}{\sqrt{F_1}} \left(\frac{F_1}{U_1}\right)^{-\varepsilon}$$

$$F_2 = G[l_1, p_1] \quad F_1 = G[l_2, l_1 + p_1] \quad U_2 = G[p_1] \quad U_1 = G[l_1 + p_1]$$

$$F_{2cut1} \sim D_5^2 - 2D_5 + F_{2cut15} \quad F_{1cut23} \sim D_5^2$$

$$\sqrt{F_1 F_2} \sim \sqrt{\mathcal{O}(D_5^4)}$$

Conclusion

With the hint from intersection theory， we construct dlog-form integrals in form (in Baikov representation):

$$I = \int_{\mathcal{C}} u\varphi$$

$$u = \prod_i p_i^{\nu_i} \quad \nu_i \in \mathbb{C} \quad \varphi = \frac{q}{\prod_i p_i^{n_i}} \bigwedge_j dz_j \quad n_i \in \mathbb{N}_0$$

we construct enough dlog Master integrals for all one-form cases without elliptic integral, and we also get their alphabet in general form.

we also construct enough dlog Master Integral for 4 mass triangle and some other cases not show here.

Outlook

- for n-form and a given u, how to know there is a dlog basis.
- Is there a way to directly construct dlog at all variables level?
- Extend to elliptic cases
- …

dlog building block

$$\frac{1}{x}dx \sim dlog(x)$$
$$\frac{\sqrt{b^2-4ac}}{ax^2+bx+c}dx \sim dlog(x-x_2) - dlog(x-x_1)$$
$$\frac{bx+2c}{x(ax^2+bx+c)}dx \sim 2dlog(x) - dlog(x-x_2) - dlog(x-x_1)$$
$$\frac{\sqrt{a}}{\sqrt{ax^2+bx+c}}dx \sim dlog(\frac{\sqrt{(x-x_1)}-\sqrt{(x-x_2)}}{\sqrt{(x-x_1)}+\sqrt{(x-x_2)}})$$

$$\frac{\sqrt{c}}{x\sqrt{ax^2+bx+c}}dx \sim \frac{\sqrt{c}}{\sqrt{ct^2+bt+a}}dt \quad t = \frac{1}{x}$$

some UT function:

$$G^{\varepsilon}$$

$$\Gamma[1-\varepsilon]$$

$$\frac{\Gamma[\frac{1}{2}-\varepsilon]}{\sqrt{\pi}}$$

Differential equation for 2 sqrt case:

u= $(x-\text{a1})^{\text{eps}}\,(x-\text{a2})^{\text{eps}}\,(x-\text{a3})^{\text{eps}-\frac{1}{2}}\,(x-\text{a4})^{\text{eps}-\frac{1}{2}}$

$$\langle \frac{d}{ds_i}\varphi_j + \frac{dlog(u(z))}{ds_i}\varphi_j|\varphi_k\rangle \,.\boldsymbol{C}^{-1}_{km}$$

$$de_i = \varepsilon\; dA_{ij}\; e_j$$

A= $\begin{pmatrix} 4\log(\text{a3}-\text{a4}) & -2\left(\sinh^{-1}\left(\sqrt{\frac{\text{a1}-\text{a3}}{\text{a3}-\text{a4}}}\right)+\tanh^{-1}\left(\sqrt{\frac{\text{a1}-\text{a3}}{\text{a1}-\text{a4}}}\right)\right) & -2\left(\sinh^{-1}\left(\sqrt{\frac{\text{a2}-\text{a3}}{\text{a3}-\text{a4}}}\right)+\tanh^{-1}\left(\sqrt{\frac{\text{a2}-\text{a3}}{\text{a2}-\text{a4}}}\right)\right) \\ 8\left(\sinh^{-1}\left(\sqrt{\frac{\text{a1}-\text{a3}}{\text{a3}-\text{a4}}}\right)+\tanh^{-1}\left(\sqrt{\frac{\text{a1}-\text{a3}}{\text{a1}-\text{a4}}}\right)\right) & \log(\text{a1}-\text{a2})+2\log(\text{a1}-\text{a3})+2\log(\text{a1}-\text{a4})-\log(\text{a3}-\text{a4}) & 2\left(\tanh^{-1}\left(\sqrt{\frac{(\text{a1}-\text{a4})(\text{a2}-\text{a3})}{(\text{a1}-\text{a3})(\text{a2}-\text{a4})}}\right)+\tanh^{-1}\left(\sqrt{\frac{(\text{a1}-\text{a3})(\text{a2}-\text{a4})}{(\text{a1}-\text{a4})(\text{a2}-\text{a3})}}\right)\right) \\ 8\left(\sinh^{-1}\left(\sqrt{\frac{\text{a2}-\text{a3}}{\text{a3}-\text{a4}}}\right)+\tanh^{-1}\left(\sqrt{\frac{\text{a2}-\text{a3}}{\text{a2}-\text{a4}}}\right)\right) & 2\left(\tanh^{-1}\left(\sqrt{\frac{(\text{a1}-\text{a4})(\text{a2}-\text{a3})}{(\text{a1}-\text{a3})(\text{a2}-\text{a4})}}\right)+\tanh^{-1}\left(\sqrt{\frac{(\text{a1}-\text{a3})(\text{a2}-\text{a4})}{(\text{a1}-\text{a4})(\text{a2}-\text{a3})}}\right)\right) & \log(\text{a1}-\text{a2})+2\log(\text{a2}-\text{a3})+2\log(\text{a2}-\text{a4})-\log(\text{a3}-\text{a4}) \end{pmatrix}$

Triangle 4 mass c4 c5

$$\hat{\varphi}_4^{(1)}(\boldsymbol{z}) = \hat{\phi}_4^{(1)}(\boldsymbol{z}) + \hat{\phi}_5^{(1)}(\boldsymbol{z}) = \frac{(z_3+m^2)\sqrt{\lambda}}{\sqrt{\rho_2}}\left[\frac{\partial \log\left((z-c_4)(z-c_5)\right)}{\partial z}\frac{\partial}{\partial z_2} - 2\frac{\partial^2}{\partial z_2 \partial z}\right][(z-c_4)(z-c_5)]$$

$$= \frac{\sqrt{\lambda}\sqrt{\rho_2}}{2(z_3+m^2)^2(c_4-z)(z-c_5)}\left[\rho_3(m_2^2-z_1+z_3)+(z_3+m^2)(c_4+c_5-2z)(2m^2-z_2+z_3+z_5)\right],$$

$$\hat{\varphi}_5^{(1)}(\boldsymbol{z}) = \hat{\phi}_4^{(1)}(\boldsymbol{z}) - \hat{\phi}_5^{(1)}(\boldsymbol{z}) = \frac{\sqrt{\lambda}\sqrt{\rho_3}}{(z-c_4)(z-c_5)}\frac{\partial}{\partial z_2}[(z-c_4)(z-c_5)]$$

$$= \frac{\sqrt{\lambda}\sqrt{\rho_3}}{2(z_3+m^2)^2(z-c_4)(z-c_5)}\left[\rho_2(2m^2-z_2+z_3+z_5)+(z_3+m^2)(c_4+c_5-2z)(m_2^2-z_1+z_3)\right],$$

$$\rho_1 = \lambda(s, z_4, z_5) - 4sm^2\,,$$
$$\rho_2 = \lambda(m_2^2, z_1, z_3) - 4m_2^2m^2\,,$$
$$\rho_3 = \lambda(z_2, z_3, z_5) - 4z_2m^2\,.$$

$\rho_3 \to 0$ in the limit $z_{2,3,5} \to 0.$

Triangle 4 mass subsector

$$\hat{\varphi}(\boldsymbol{z}) \in \left\{ \frac{\hat{\varphi}_3^{(1)}(\boldsymbol{z})}{z_1 z_2 z_3 z_5} \frac{1}{\sqrt{\rho_1}},\ \frac{\hat{\varphi}_3^{(1)}(\boldsymbol{z})}{z_1 z_2 z_3 z_4} \frac{1}{\sqrt{\rho_1}},\ \frac{\hat{\varphi}_4^{(1)}(\boldsymbol{z})}{z_1 z_2 z_4 z_5} \frac{1}{\sqrt{\rho_2}},\ \frac{\hat{\varphi}_4^{(1)}(\boldsymbol{z})}{z_2 z_3 z_4 z_5} \frac{1}{\sqrt{\rho_2}},\right.$$
$$\left. \frac{\hat{\varphi}_5^{(1)}(\boldsymbol{z})}{z_1 z_3 z_4 z_5} \frac{1}{\sqrt{\rho_3}},\ \frac{\hat{\varphi}_5^{(1)}(\boldsymbol{z})}{z_1 z_2 z_4 z_5} \frac{1}{\sqrt{\rho_3}},\ \frac{\hat{\varphi}_5^{(1)}(\boldsymbol{z})}{z_1 z_2 z_3 z_4} \frac{1}{\sqrt{\rho_3}},\ \dots \right\}.$$