Stony Brook University

A Forest-based Infrared Subtraction for Wide-angle Scattering

高能微扰QCD论坛, 2020年8月21日

Yao Ma （马尧）
C. N. Yang Institute for Theoretical Physics, Stony Brook University

Based on JHEP05(2020)012
(arXiv: 1910.11304)

Computing QCD Amplitudes

- **QCD amplitudes are necessary ingredients to all LHC cross sections.**

Computing QCD Amplitudes

- **QCD amplitudes are necessary ingredients to all LHC cross sections.**

- **A major bottleneck in precise calculations is due to the IR divergences, which arise from soft and/or collinear momentum configurations.**

Computing QCD Amplitudes

- **QCD amplitudes are necessary ingredients to all LHC cross sections.**

- **A major bottleneck in precise calculations is due to the IR divergences, which arise from soft and/or collinear momentum configurations.**

- **Infrared subtraction（红外减除）: to make the divergent integrand finite by subtracting from it a suitable counterterm whose singular behavior matches that of the original integrand.**

Computing QCD Amplitudes

- **QCD amplitudes are necessary ingredients to all LHC cross sections.**

- **A major bottleneck in precise calculations is due to the IR divergences, which arise from soft and/or collinear momentum configurations.**

- **Infrared subtraction（红外减除）: to make the divergent integrand finite by subtracting from it a suitable counterterm whose singular behavior matches that of the original integrand.**

- **Today: an all-order IR subtraction based on QCD factorization and Zimmermann's forest formula.**

Outline

- Introduction
- Approximations and induced IR divergences
- A forest formula to subtract IR divergences
- Factorization --- a byproduct
- Summary and outlook

Outline

- Introduction
    - Infrared divergences in perturbative QCD
    - The forest structure of subtractions
- Approximations and induced IR divergences
    - Definition: hard-collinear & soft-collinear
    - Induced pinch surfaces
- A forest formula to subtract IR divergences
    - Expression
    - Pairwise IR cancellation
- Factorization --- a byproduct
    - Factorization near a pinch surface
    - Hard-soft-collinear factorization to all orders
- Summary and outlook

Infrared (IR) divergences in perturbative QCD

- **In momentum space, IR divergences are due to that the infinities of the integrand dominates the smallness of the integration measure.**

Infrared (IR) divergences in perturbative QCD

- **In momentum space, IR divergences are due to that the infinities of the integrand dominates the smallness of the integration measure.**

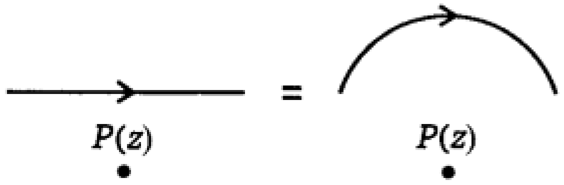
$$\underset{\substack{\uparrow\\ \text{external momenta}}}{G\left(\{p_a^\mu\}\right)} = \prod_{i=1}^{N} \int_0^1 \underset{\substack{\uparrow\\ \text{Feynman parameters}}}{d\alpha_i}\, \delta\left(\sum_i \alpha_i - 1\right) \prod_b \int \underset{\substack{\uparrow\\ \text{loop momenta}}}{\frac{d^D k_b}{(2\pi)^D}} \frac{N\left(p_a, k_b\right)}{\left[D\left(p_a, k_b; \alpha_i\right)\right]^n}$$

with $D\left(p_a, k_b; \alpha_i\right) \equiv \sum_{j=1}^{N} \alpha_j l_j^2(p_a, k_b) + i\epsilon$

- $D\left(p_a, k_b; \alpha_i\right)$ **must vanish if** $G\left(\{p_a^\mu\}\right)$ **is IR divergent.**

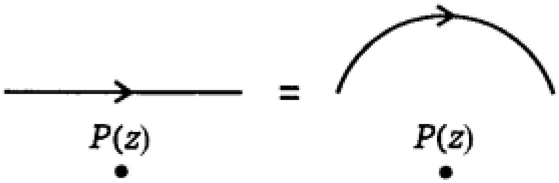
Infrared (IR) divergences in perturbative QCD

- $D\left(p_a, k_b; \alpha_i\right) = 0$ **is insufficient for an IR divergence.**



Infrared (IR) divergences in perturbative QCD

- $D\left(p_a, k_b; \alpha_i\right) = 0$ **is insufficient for an IR divergence.**

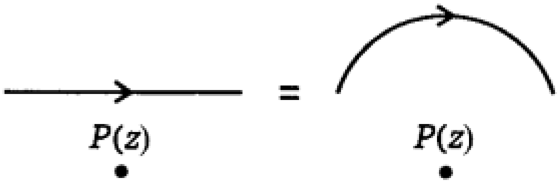


- **There are two exceptions where contours cannot be deformed:**

  (1) endpoint singularity

Infrared (IR) divergences in perturbative QCD

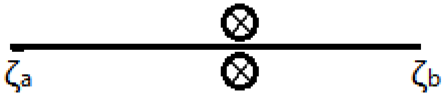
- $D\left(p_a, k_b; \alpha_i\right) = 0$ **is insufficient for an IR divergence.**



- **There are two exceptions where contours cannot be deformed:**

(1) endpoint singularity

(2) pinch singularity



Infrared (IR) divergences in perturbative QCD

- **To formalize the singularities → Landau equations （朗道方程）** **(a necessary condition for IR divergence)**

$$\begin{cases} D\left(p_a, k_b; \alpha_i\right) = 0, \\ \frac{\partial}{\partial k_J^\mu} D\left(p_a, k_b; \alpha_i\right) = 0, \quad \forall j \end{cases}$$

where $D\left(p_a, k_b; \alpha_i\right) \equiv \sum_{j=1}^{N} \alpha_j l_j^2(p_a, k_b) + i\epsilon$

Infrared (IR) divergences in perturbative QCD

- **To formalize the singularities → Landau equations （朗道方程）**
  (a necessary condition for IR divergence)

$$\begin{cases} D\left(p_a, k_b; \alpha_i\right) = 0, \\ \frac{\partial}{\partial k_J^\mu} D\left(p_a, k_b; \alpha_i\right) = 0, \quad \forall j \end{cases}$$

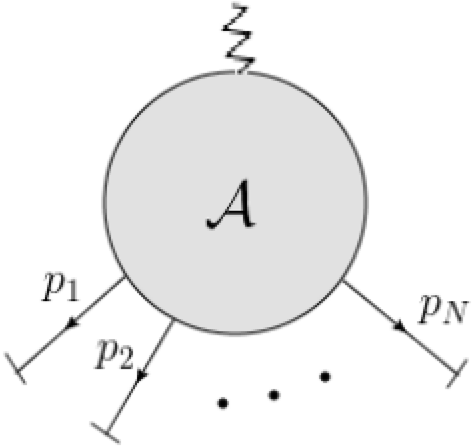
where $D\left(p_a, k_b; \alpha_i\right) \equiv \sum_{j=1}^{N} \alpha_j l_j^2(p_a, k_b) + i\epsilon$

- **These equations lead to a classical picture of any given IR divergent region**
  --- the "pinch surface".

Stony Brook University

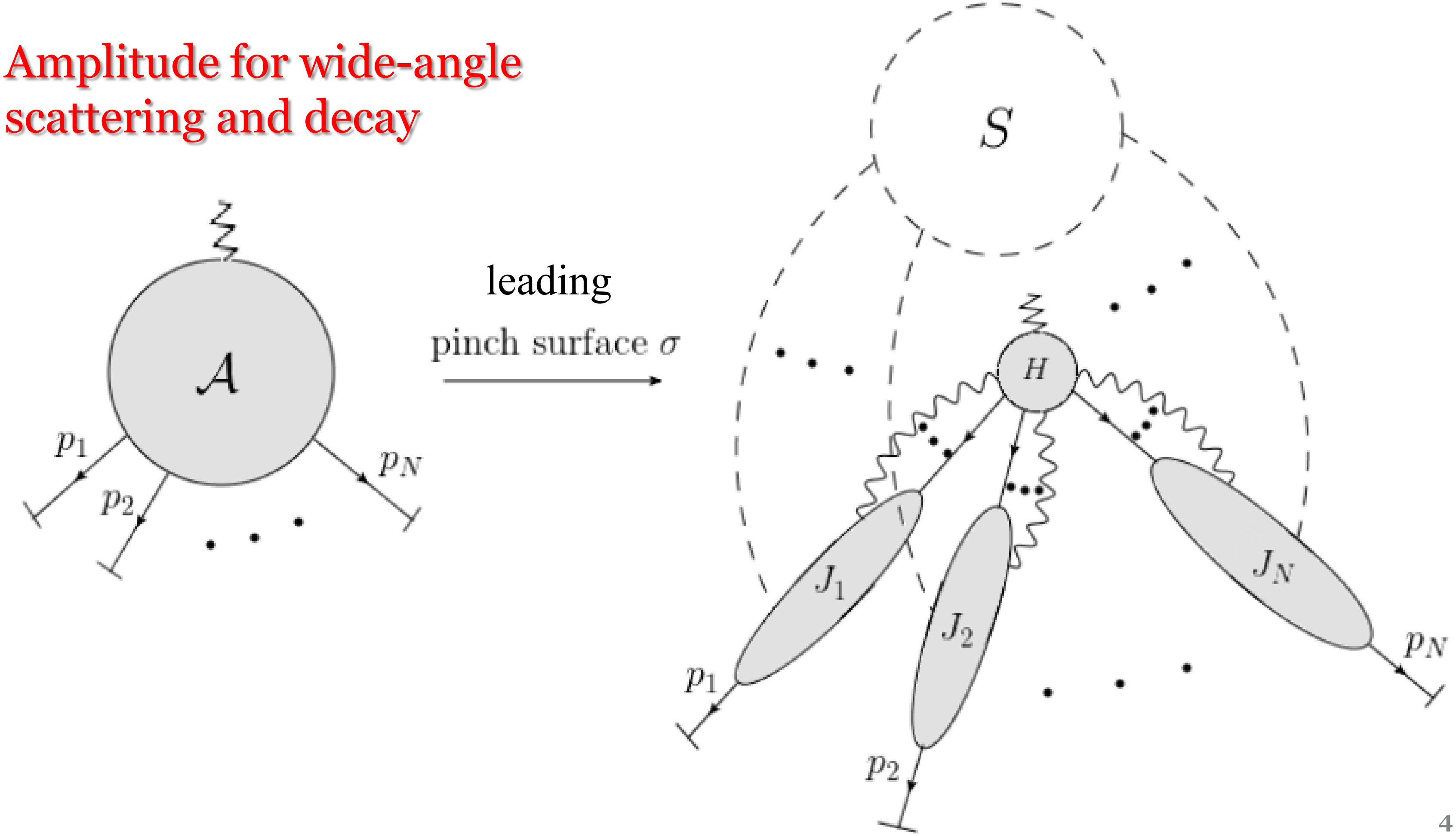
Result: the general picture

Amplitude for wide-angle scattering and decay



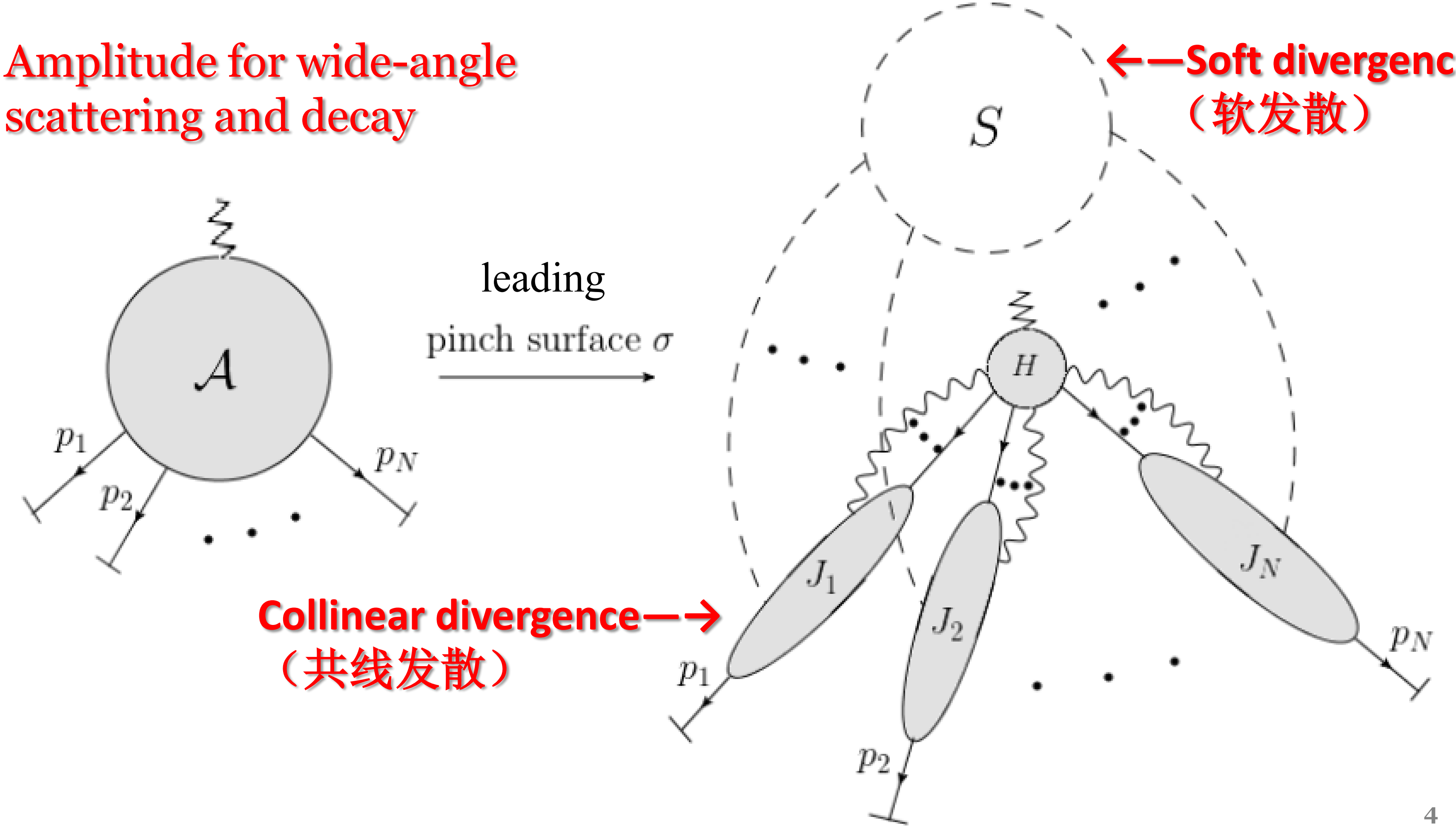
Result: the general picture

Amplitude for wide-angle scattering and decay



Result: the general picture

Amplitude for wide-angle scattering and decay



←—Soft divergence
（软发散）

Collinear divergence—→
（共线发散）

Outline

- Introduction
    - Infrared divergences in perturbative QCD
    - The forest structure of subtractions
- Approximations and induced IR divergences
    - Definition: hard-collinear & soft-collinear
    - Induced pinch surfaces
- A forest formula to subtract IR divergences
    - Expression
    - Pairwise IR cancellation
- Factorization --- a byproduct
    - Factorization near a pinch surface
    - Hard-soft-collinear factorization to all orders
- Summary and outlook

Approximation operator t_σ

For each given Feynman diagram, t_σ only acts on the Feynman integrand.

For each jet with three-velocity $\mathbf{v}_A$, we define

$$\beta_A^\mu = \frac{1}{\sqrt{2}}\left(1, \mathbf{v}_A\right), \quad \overline{\beta}_A^\mu = \frac{1}{\sqrt{2}}\left(1, -\mathbf{v}_A\right)$$

Approximation operator t_σ

$$\beta_A^\mu = \frac{1}{\sqrt{2}}(1, \mathbf{v}_A)\,, \quad \overline{\beta}_A^\mu = \frac{1}{\sqrt{2}}(1, -\mathbf{v}_A)$$

- Hard-collinear approximation（硬共线近似）

$$H^{(\sigma)}\left(p^\mu - \sum_i k_i^\mu, \{k_i^{\alpha_i}\}\right)_\eta^{\{\mu_i\}} \xrightarrow{\mathrm{hc}_A} H^{(\sigma)}\left(\left((p - \sum_i k_i) \cdot \overline{\beta}_A\right)\beta_A^\mu, \{\left(k_i \cdot \overline{\beta}_A\right)\beta_A^{\alpha_i}\}\right)_{\{\nu_i\},\eta}$$

jet momenta

$$\cdot \prod_j \beta_A^{\nu_j} \overline{\beta}_A^{\mu_j} \cdot \begin{cases} \frac{1}{2}(\gamma \cdot \beta_A)(\gamma \cdot \overline{\beta}_A) & \text{fermion line,} \\ 1 & \text{otherwise, etc.} \end{cases}$$

- Soft-collinear approximation （软共线近似）

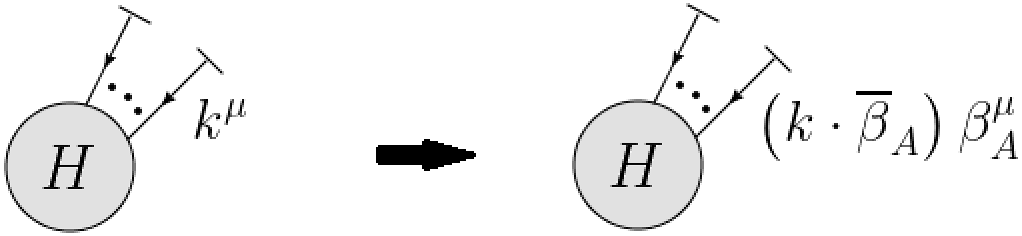
$$J_A^{(\sigma)}(\{l_i^{\alpha_i}\})_\eta^{\{\mu_i\}} \xrightarrow{\mathrm{sc}_A} J_A^{(\sigma)}\left(\left\{(l_i \cdot \beta_A)\,\overline{\beta}_A^{\alpha_i}\right\}\right)_{\{\nu_i\},\eta} \prod_j \beta_A^{\nu_j} \overline{\beta}_A^{\mu_j}\,.$$

soft momenta

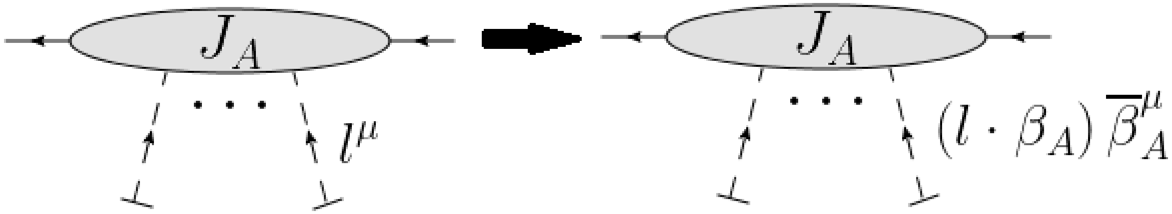
Approximation operator t_σ

$$\beta_A^\mu = \frac{1}{\sqrt{2}}(1, \mathbf{v}_A)\,, \quad \overline{\beta}_A^\mu = \frac{1}{\sqrt{2}}(1, -\mathbf{v}_A)$$

- Hard-collinear approximation （硬共线近似）



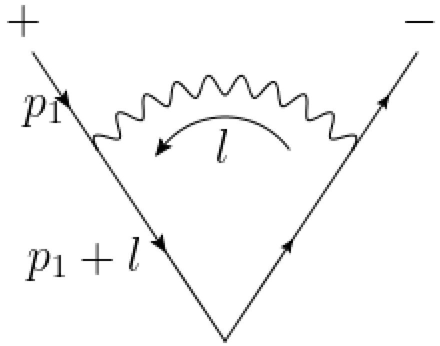
- Soft-collinear approximation （软共线近似）



▸ These approximations ONLY act on the Feynman integrand.

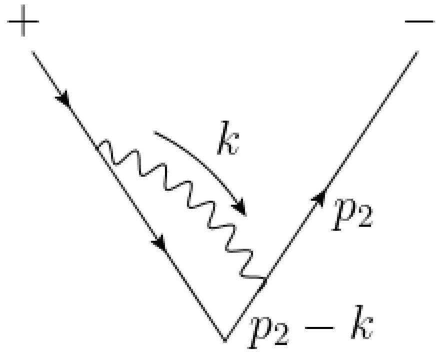
Lowest order examples (backup)

Soft



$$(p_1 + l)^2 \sim p_1^2 + 2p_1^+ l^-$$

Collinear



$$(p_2 - k)^2 \sim -2p_2^- k^+$$

Induced pinch surfaces

- **Motivation**

$$\mathcal{A} - t_\sigma \mathcal{A} + \cdots$$

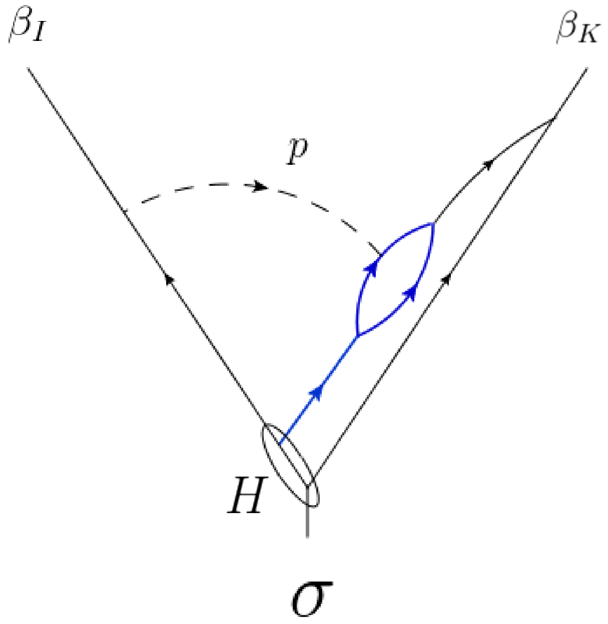
The subtraction term $-\, t_\sigma \mathcal{A}$ cancels the IR divergence at σ ;

Meanwhile, this term may induce some new IR regions.

Induced pinch surfaces

- **The pinch surfaces of $t_\sigma \mathcal{A}$ could be very different from those of $\mathcal{A}$.**
  A systematic study is needed.

- **For example,**



$$\beta_I^\mu \equiv \frac{1}{\sqrt{2}}(1, \mathbf{v}_I)$$

$$\beta_K^\mu \equiv \frac{1}{\sqrt{2}}(1, \mathbf{v}_K)$$

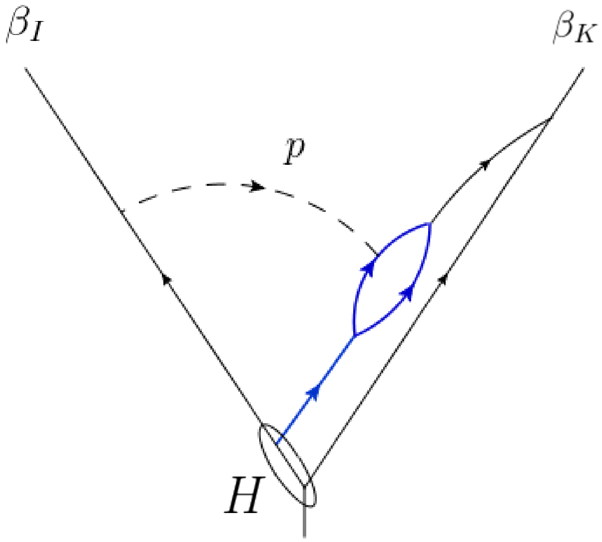
They may not be back-to-back.

What can a pinch surface of $t_\sigma \mathcal{A}$ look like?

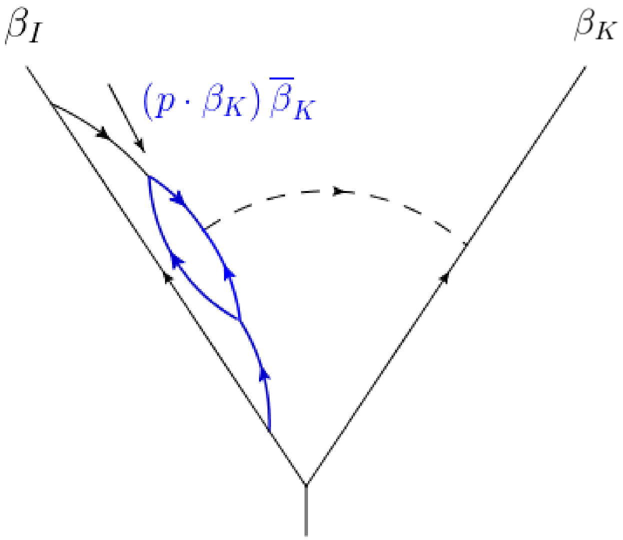
Induced pinch surfaces

- **The pinch surfaces of $t_\sigma \mathcal{A}$ could be very different from those of $\mathcal{A}$.**
  A systematic study is needed.

- **For example,**



σ

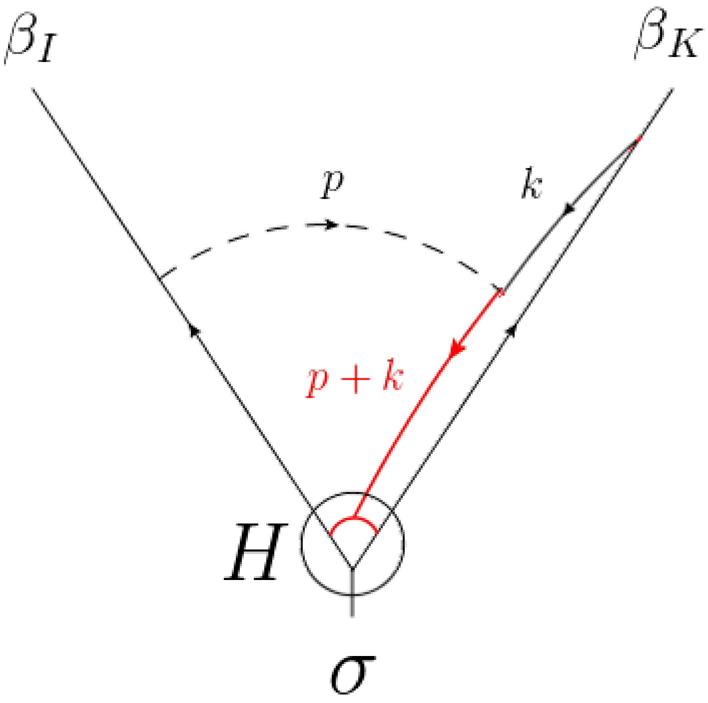


a pinch surface of $t_\sigma \mathcal{A}$

$\rho^{\{\sigma\}}$

Induced pinch surfaces

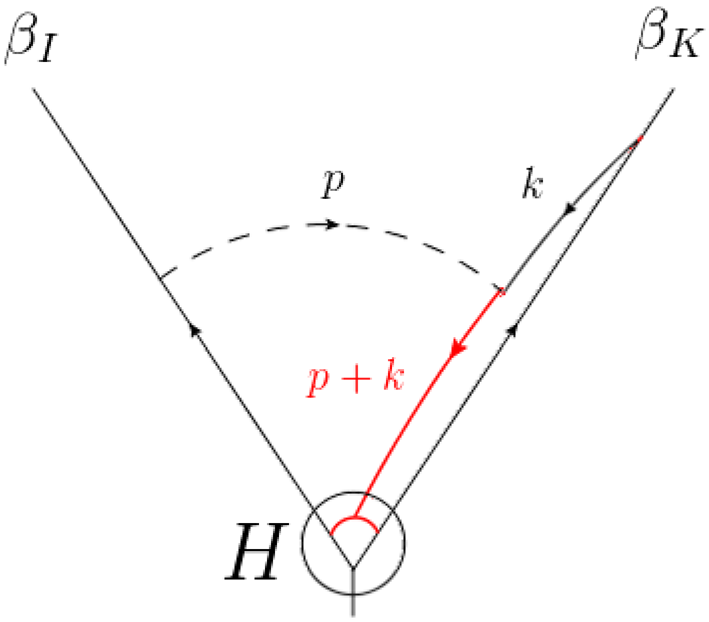
- **Another example:**



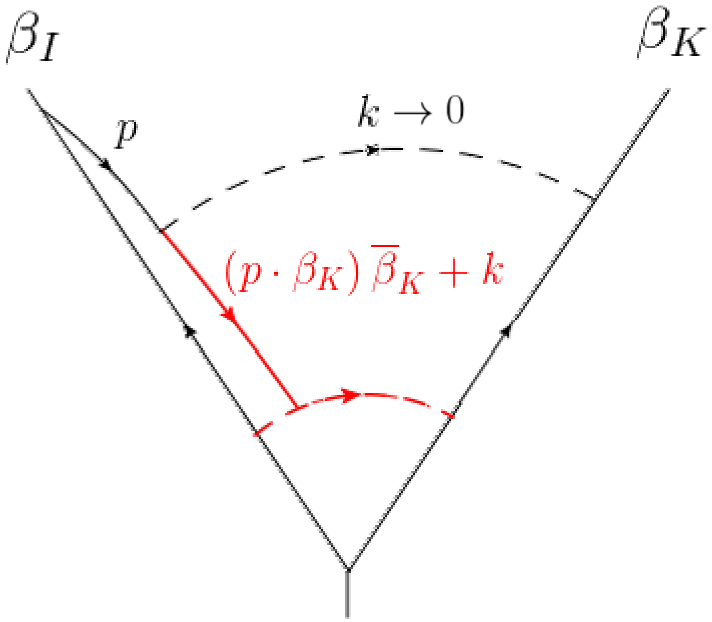
What can a pinch surface of $t_\sigma \mathcal{A}$ look like?

Induced pinch surfaces

- **Another example:**



σ



a pinch surface of $t_\sigma \mathcal{A}$

$\rho^{\{\sigma\}}$

Induced pinch surfaces

- **An all-order result:**

(*Theorem 1*) In the induced pinch surface $\rho^{\{\sigma\}}$, all the propagators of $J_I^{(\sigma)}$ that are lightlike can only be collinear to β_I^μ or $\overline{\beta}_I^\mu$.

- **The direction(s) of the subdiagram $J_I^{(\sigma)} \bigcap J_K^{(\rho^{\{\sigma\}})}$ is fixed!**

Comments on $t_\sigma \mathcal{A}$

- **The pinch surfaces of $t_\sigma \mathcal{A}$ could be very different from those of $\mathcal{A}$.**

  A systematic study has been given in the paper.

- **However, the IR divergences in those “unphysical” regions are still at worst logarithmic.**

  A power counting technique is involved.

- **The approximation operators $t_{\sigma_1}, t_{\sigma_2}, \ldots, t_{\sigma_n}$ can act repetitively on $\mathcal{A}$.**

  The pinch surfaces $\sigma_1, \sigma_2, \ldots, \sigma_n$ should be ordered.

$$t_{\sigma_1} t_{\sigma_2} \ldots t_{\sigma_n} \mathcal{A}$$

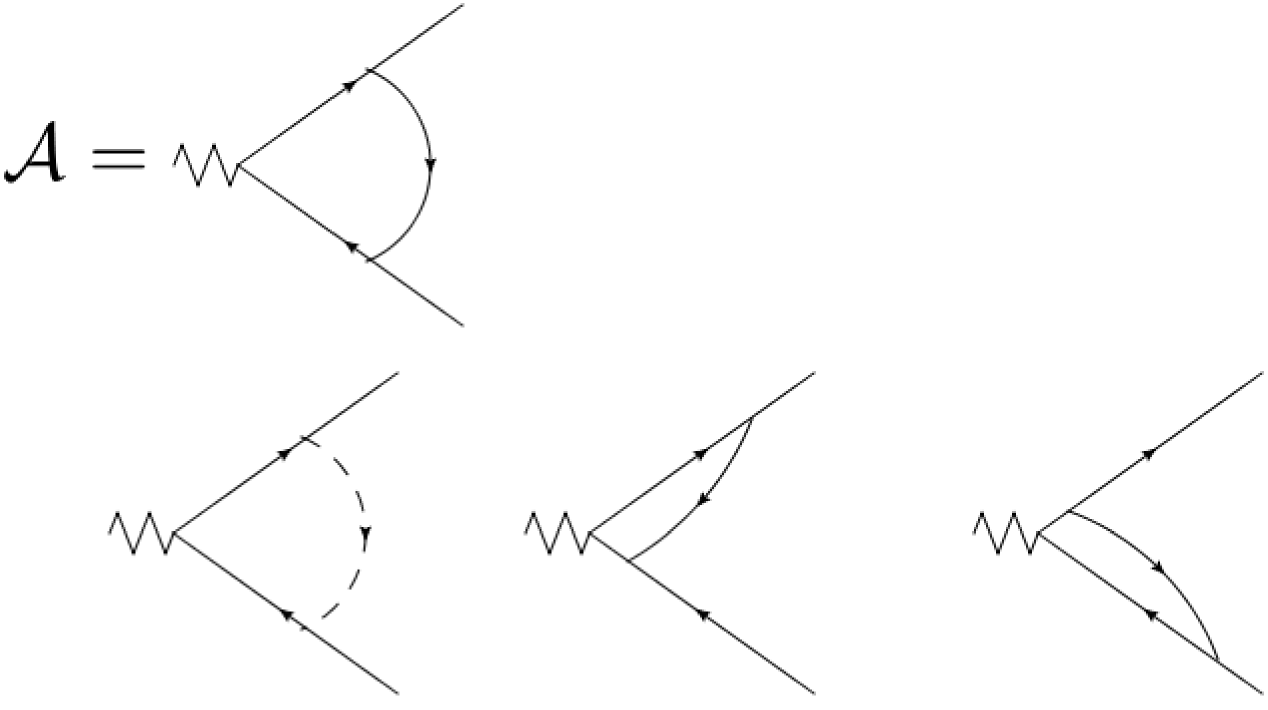
Outline

- Introduction
    - Infrared divergences in perturbative QCD
    - The forest structure of subtractions
- Approximations and induced IR divergences
    - Definition: hard-collinear & soft-collinear
    - Induced pinch surfaces
- A forest formula to subtract IR divergences
    - Expression
    - Pairwise IR cancellation
- Factorization --- a byproduct
    - Factorization near a pinch surface
    - Hard-soft-collinear factorization to all orders
- Summary and outlook

To order the pinch surfaces

(From the “smallest” to the “largest”)

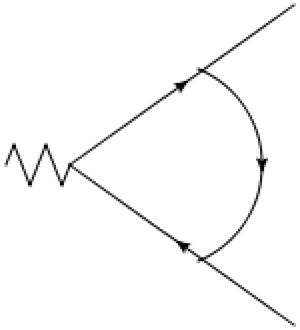
To order the pinch surfaces

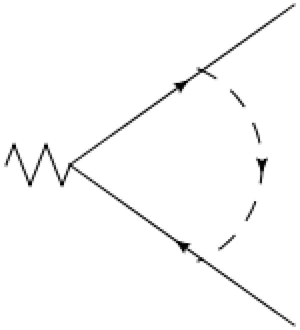


σ_1 σ_2 σ_3

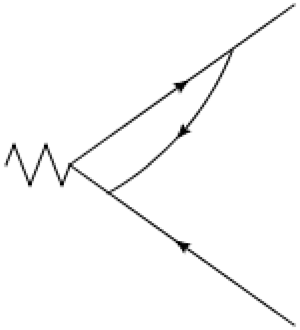
To order the pinch surfaces

$\mathcal{A} =$

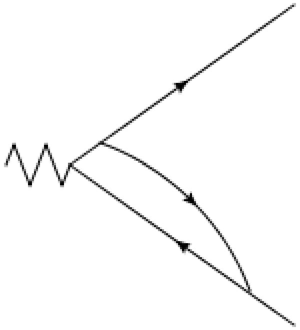




σ_1



σ_2



σ_3

$$\sigma_1 \subset \sigma_2, \quad \sigma_1 \subset \sigma_3$$

To order the pinch surfaces

- **More precisely, we should develop the concept of "normal space" for loop momentum k^μ in pinch surface σ :**

$$\mathcal{N}_\sigma\left(k^\mu\right) \equiv \begin{cases} \varnothing \text{ (empty)} & \text{if } k^\mu \text{ is hard in } \sigma, \\ \text{span}\left\{\overline{\beta}^\mu,\ \beta_\perp^\mu\right\} & \text{if } k^\mu \text{ is collinear to } \beta^\mu \text{ in } \sigma, \\ \text{the full 4-dim space} & \text{if } k^\mu \text{ is soft in } \sigma. \end{cases}$$

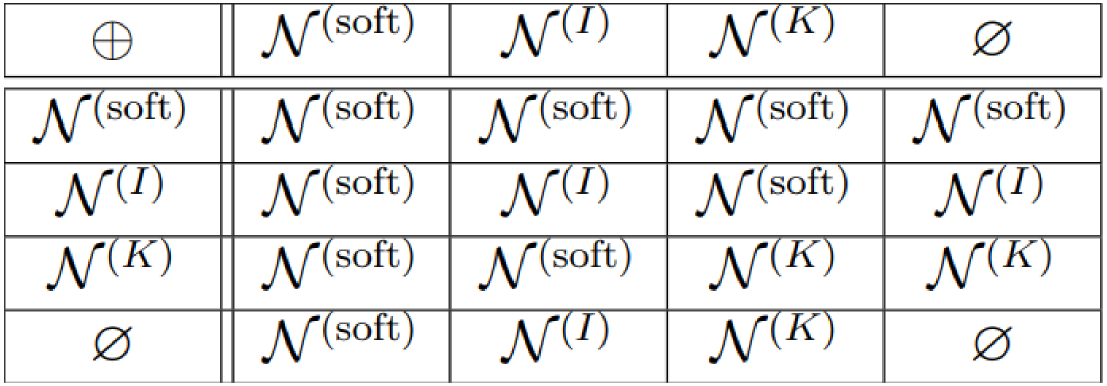
Then,

$$\sigma_1 \subset \sigma_2 \Leftrightarrow \mathcal{N}_{\sigma_1}\left(k_i\right) \supseteq \mathcal{N}_{\sigma_2}\left(k_i\right),\ \forall \text{ loop momentum } k_i^\mu$$

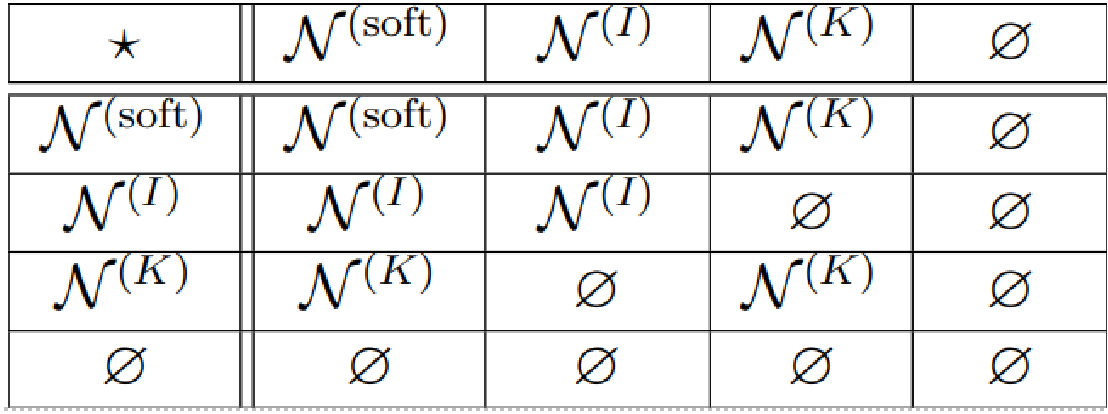
(nested，嵌套)

Normal space algebra

- "Plus" and "star" operations
  (union) & (intersection)



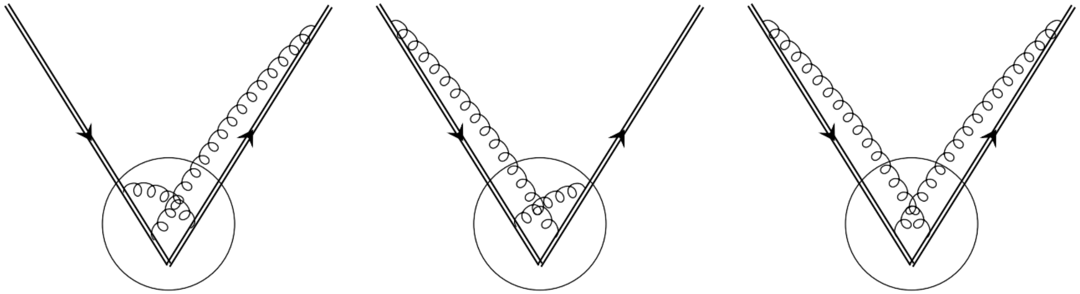
$\oplus$	$\mathcal{N}^{(\mathrm{soft})}$	$\mathcal{N}^{(I)}$	$\mathcal{N}^{(K)}$	$\varnothing$
$\mathcal{N}^{(\mathrm{soft})}$	$\mathcal{N}^{(\mathrm{soft})}$	$\mathcal{N}^{(\mathrm{soft})}$	$\mathcal{N}^{(\mathrm{soft})}$	$\mathcal{N}^{(\mathrm{soft})}$
$\mathcal{N}^{(I)}$	$\mathcal{N}^{(\mathrm{soft})}$	$\mathcal{N}^{(I)}$	$\mathcal{N}^{(\mathrm{soft})}$	$\mathcal{N}^{(I)}$
$\mathcal{N}^{(K)}$	$\mathcal{N}^{(\mathrm{soft})}$	$\mathcal{N}^{(\mathrm{soft})}$	$\mathcal{N}^{(K)}$	$\mathcal{N}^{(K)}$
$\varnothing$	$\mathcal{N}^{(\mathrm{soft})}$	$\mathcal{N}^{(I)}$	$\mathcal{N}^{(K)}$	$\varnothing$



$\star$	$\mathcal{N}^{(\mathrm{soft})}$	$\mathcal{N}^{(I)}$	$\mathcal{N}^{(K)}$	$\varnothing$
$\mathcal{N}^{(\mathrm{soft})}$	$\mathcal{N}^{(\mathrm{soft})}$	$\mathcal{N}^{(I)}$	$\mathcal{N}^{(K)}$	$\varnothing$
$\mathcal{N}^{(I)}$	$\mathcal{N}^{(I)}$	$\mathcal{N}^{(I)}$	$\varnothing$	$\varnothing$
$\mathcal{N}^{(K)}$	$\mathcal{N}^{(K)}$	$\varnothing$	$\mathcal{N}^{(K)}$	$\varnothing$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

Normal space algebra

- **These operations enable us to evaluate the “intersection” of two given pinch surfaces:**



(a) (b) (c)

(Erdogan & Sterman 2015)

In this example, (c) is the intersection of (a) and (b)!

“enclosed pinch surface”

$$\mathcal{N}_{\mathrm{enc}\left[\sigma,\rho^{\{\sigma\}}\right]}\left(l^{\mu}\right) = \mathcal{N}_{\sigma}\left(l^{\mu}\right) \oplus \mathcal{N}_{\rho^{\{\sigma\}}}\left(l^{\mu}\right) .$$

Forest Formula

$$\left[\sum_{F\in\mathcal{F}[\mathcal{A}]}\prod_{\sigma\in F}(-t_\sigma)\,\mathcal{A}\right]_{\mathrm{div}}=0.$$

Forest Formula $$\left[\sum_{F\in\mathcal{F}[\mathcal{A}]}\prod_{\sigma\in F}(-t_\sigma)\,\mathcal{A}\right]_{\rm div}=0.$$

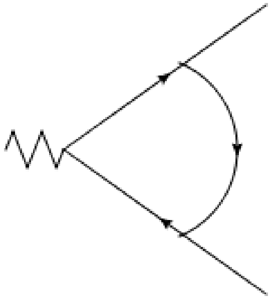
- Each F is called a "forest": a set of nested pinch surfaces.

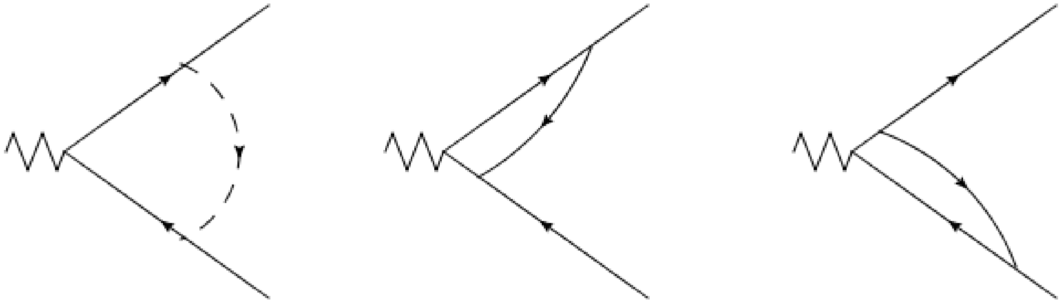
- t_σ is the approximation operator, whose associated pinch surface σ is an element of the forest F.

- The t_σ products are ordered (smallest pinch surface to the right).

An example of $\left[\sum_{F\in\mathcal{F}[\mathcal{A}]}\prod_{\sigma\in F}\left(-t_\sigma\right)\mathcal{A}\right]_{\mathrm{div}} = 0.$

$\mathcal{A} =$

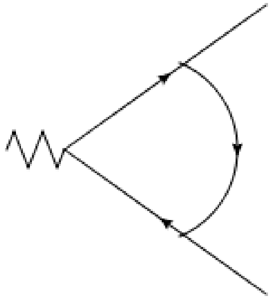


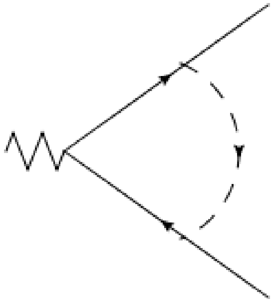


σ_1 σ_2 σ_3

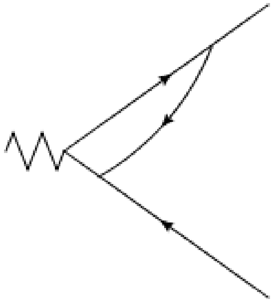
An example of $\left[\sum_{F\in\mathcal{F}[\mathcal{A}]}\prod_{\sigma\in F}\left(-t_\sigma\right)\mathcal{A}\right]_{\text{div}} = 0.$

$\mathcal{A} =$

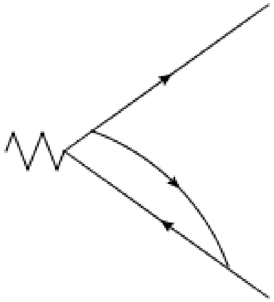




σ_1



σ_2

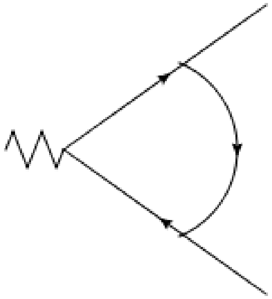


σ_3

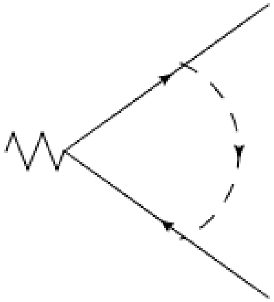
$$F = \varnothing, \{\sigma_1\}, \{\sigma_2\}, \{\sigma_3\}, \{\sigma_1, \sigma_2\}, \{\sigma_1, \sigma_3\}$$

An example of $\left[\sum_{F\in\mathcal{F}[\mathcal{A}]}\prod_{\sigma\in F}\left(-t_\sigma\right)\mathcal{A}\right]_{\mathrm{div}}=0.$

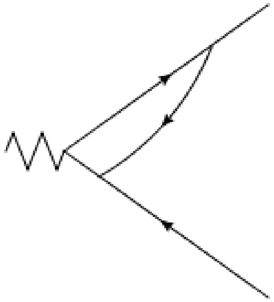
$\mathcal{A} =$



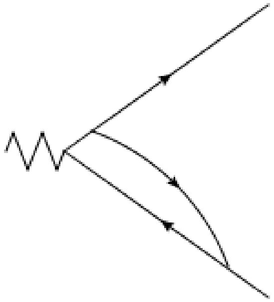
J. Collins, *Foundations of Perturbative QCD*, Cambridge University Press, 2011



σ_1



σ_2



σ_3

$$F=\varnothing,\{\sigma_1\},\{\sigma_2\},\{\sigma_3\},\{\sigma_1,\sigma_2\},\{\sigma_1,\sigma_3\}$$

$$\left(\mathcal{A}-t_{\sigma_1}\mathcal{A}-t_{\sigma_2}\mathcal{A}-t_{\sigma_3}\mathcal{A}+t_{\sigma_1}t_{\sigma_2}\mathcal{A}+t_{\sigma_1}t_{\sigma_3}\mathcal{A}\right)_{\mathrm{div}}=0$$

History: the UV Forest Formula (BPHZ)

- Bogoliubov's recursive R-operation (Bogoliubov & Parasiuk 1957)

$$R(\Gamma) = \sum_{S \subseteq \Gamma} Z(S) * \Gamma/S \,, \qquad Z(S) = \prod_{\gamma \in S} Z(\gamma) \,.$$

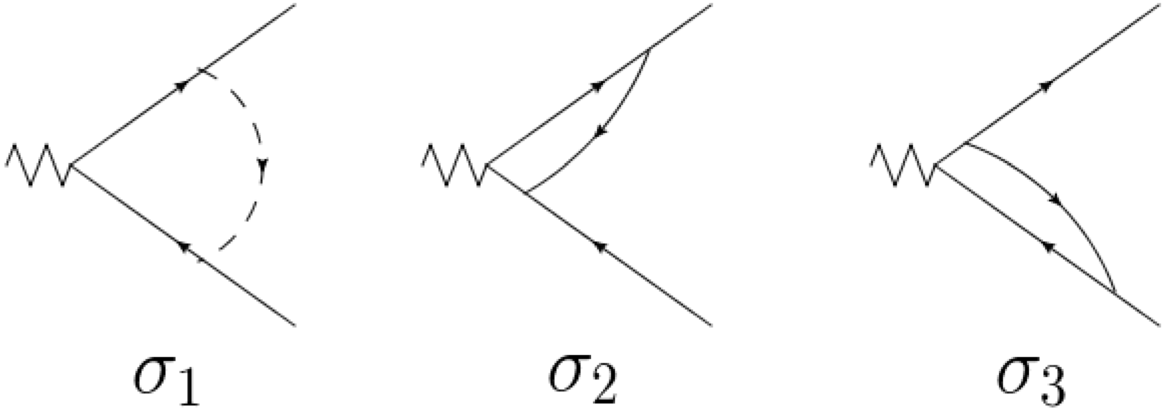
- A complete given by Hepp (Hepp 1966)
  - ▷ It is from the aspect of axiomatic QFT.
  - ▷ Overlapping divergences are avoided.

- An alternative proof by Zimmermann (Zimmermann 1968 & 1969)
  - ▷ The R-operation gives rise to a sum over "forests of graphs".

$$R_\gamma(p,k) = S_\gamma \sum_{U \in F_\gamma} \prod_{\lambda \in U} (-t_{p^\lambda}^{d(\lambda)} S_\lambda) I_\gamma(U)$$

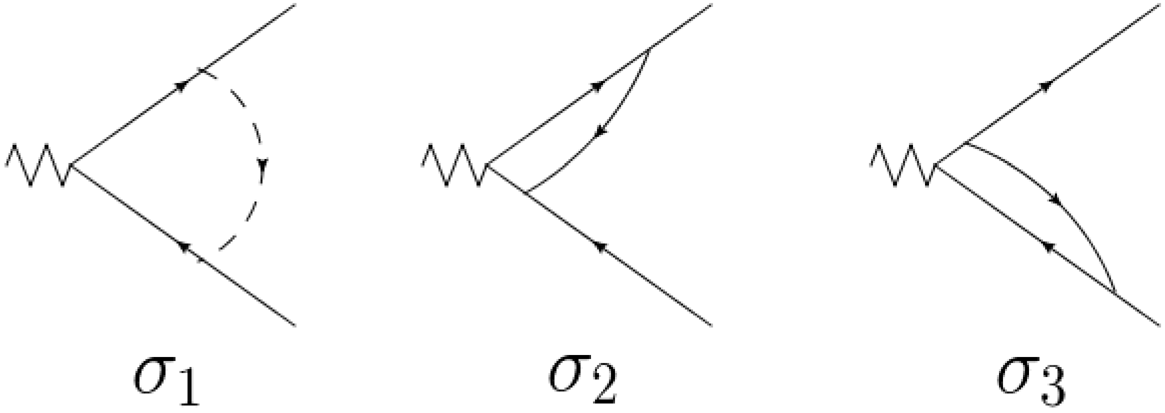
- There has been many extensions of BPHZ.

Pairwise IR cancellation



$$(\mathcal{A} - t_{\sigma_1}\mathcal{A} - t_{\sigma_2}\mathcal{A} - t_{\sigma_3}\mathcal{A} + t_{\sigma_1}t_{\sigma_2}\mathcal{A} + t_{\sigma_1}t_{\sigma_3}\mathcal{A})_{\mathrm{div}} = 0$$

Pairwise IR cancellation



$$(\mathcal{A} - t_{\sigma_1}\mathcal{A} - t_{\sigma_2}\mathcal{A} - t_{\sigma_3}\mathcal{A} + t_{\sigma_1}t_{\sigma_2}\mathcal{A} + t_{\sigma_1}t_{\sigma_3}\mathcal{A})_{\rm div} = 0$$

- **Nested divergence cancellation（嵌套发散）**
  - The term $-t_{\sigma_1}\mathcal{A}$ has a divergence at σ_2 , which is cancelled by the term $t_{\sigma_1}t_{\sigma_2}\mathcal{A}$ --- relatively simpler.

Pairwise IR cancellation

- **Overlapping divergence cancellation（交叠发散）**
  - The term $-t_{\sigma_2}\mathcal{A}$ has a divergence at σ_3, which is cancelled by the term $t_{\sigma_1}t_{\sigma_2}\mathcal{A}$.

Pairwise IR cancellation

- **Overlapping divergence cancellation（交叠发散）**
  - The term $-t_{\sigma_2}\mathcal{A}$ has a divergence at σ_3, which is cancelled by the term $t_{\sigma_1}t_{\sigma_2}\mathcal{A}$.
  - In general, an additional construction is involved.

"enclosed pinch surface"

$$\left(t_\sigma \mathcal{A}\right)_{\rho^{\{\sigma\}}} \neq 0 \qquad \tau \equiv \mathrm{enc}\left[\sigma, \rho^{\{\sigma\}}\right]$$

$$\left(t_\tau t_\sigma \mathcal{A} - t_\sigma \mathcal{A}\right)_{\mathrm{div}\ \rho^{\{\sigma\}}} = \left[\left(t_\tau - 1\right) t_\sigma \mathcal{A}\right]_{\mathrm{div}\ \rho^{\{\sigma\}}} = 0$$

To construct an enclosed pinch surface

$$\tau \equiv \mathrm{enc}\left[\sigma, \rho^{\{\sigma\}}\right]$$

- ***Theorem 3***

$$H^{(\tau)} = H^{(\sigma)} \bigcap H^{(\rho\{\sigma\})},$$

$$J_I^{(\tau)} = \left(J_I^{(\sigma)} \bigcap H^{(\rho\{\sigma\})}\right) \bigcup \left(H^{(\sigma)} \bigcap J_I^{(\rho\{\sigma\})}\right) \bigcup \left(J_I^{(\sigma)} \bigcap J_I^{(\rho\{\sigma\})}\right),$$

$$S^{(\tau)} = S^{(\sigma)} \bigcup S^{(\rho\{\sigma\})} \bigcup \left(\bigcup_{K \neq I} \left(J_I^{(\sigma)} \bigcap J_K^{(\rho\{\sigma\})}\right)\right),$$

To construct an enclosed pinch surface $\tau \equiv \mathrm{enc}\left[\sigma, \rho^{\{\sigma\}}\right]$

- ***Theorem 3***

$$H^{(\tau)} = H^{(\sigma)} \bigcap H^{(\rho\{\sigma\})},$$

$$J_I^{(\tau)} = \left(J_I^{(\sigma)} \bigcap H^{(\rho\{\sigma\})}\right) \bigcup \left(H^{(\sigma)} \bigcap J_I^{(\rho\{\sigma\})}\right) \bigcup \left(J_I^{(\sigma)} \bigcap J_I^{(\rho\{\sigma\})}\right),$$

$$S^{(\tau)} = S^{(\sigma)} \bigcup S^{(\rho\{\sigma\})} \bigcup \left(\bigcup_{K \neq I} \left(J_I^{(\sigma)} \bigcap J_K^{(\rho\{\sigma\})} \right) \right),$$

The subdiagrams of τ can be directly "read" from those of σ and $\rho^{\{\sigma\}}$.

To construct an enclosed pinch surface

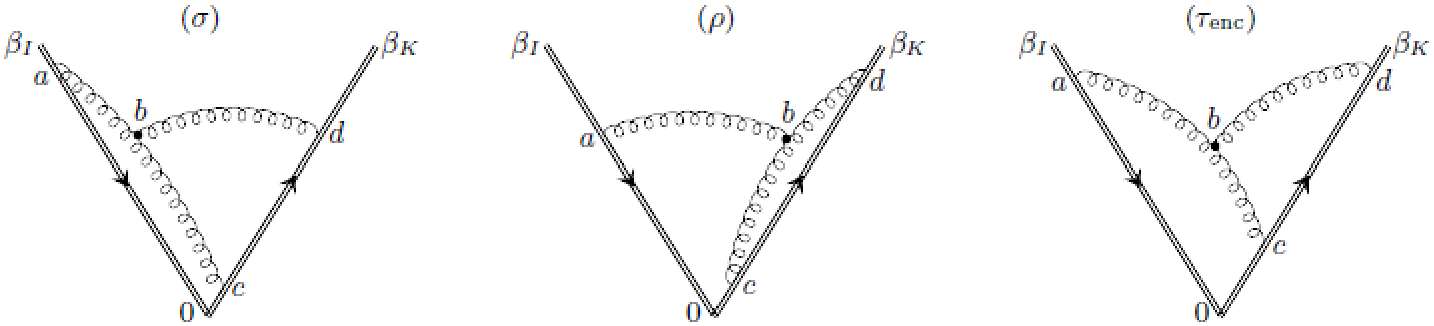
$$\tau \equiv \mathrm{enc}\left[\sigma, \rho^{\{\sigma\}}\right]$$

- ***Theorem 3***

$$H^{(\tau)} = H^{(\sigma)} \bigcap H^{\left(\rho^{\{\sigma\}}\right)},$$

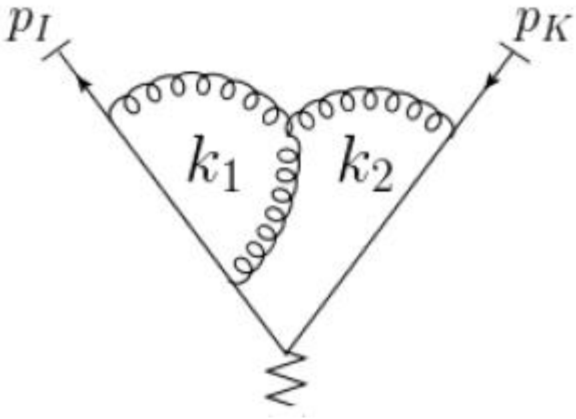
$$J_I^{(\tau)} = \left(J_I^{(\sigma)} \bigcap H^{\left(\rho^{\{\sigma\}}\right)}\right) \bigcup \left(H^{(\sigma)} \bigcap J_I^{\left(\rho^{\{\sigma\}}\right)}\right) \bigcup \left(J_I^{(\sigma)} \bigcap J_I^{\left(\rho^{\{\sigma\}}\right)}\right),$$

$$S^{(\tau)} = S^{(\sigma)} \bigcup S^{\left(\rho^{\{\sigma\}}\right)} \bigcup \left(\bigcup_{K \neq I} \left(J_I^{(\sigma)} \bigcap J_K^{\left(\rho^{\{\sigma\}}\right)} \right) \right),$$



σ $\rho^{\{\sigma\}}$ τ

A two-loop example



σ_1 (SS),	if k_1^μ and k_2^μ are both soft;
σ_2 ($\mathrm{C_1S}$),	if k_1^μ is collinear to β_I^μ and k_2^μ is soft;
σ_3 ($\mathrm{SC_2}$),	if k_1^μ is soft and k_2^μ is collinear to β_K^μ;
σ_4 ($\mathrm{C_1C_1}$),	if k_1^μ and k_2^μ are both collinear to β_I^μ;
σ_5 ($\mathrm{C_2C_2}$),	if k_1^μ and k_2^μ are both collinear to β_K^μ;
σ_6 ($\mathrm{C_1C_2}$),	if k_1^μ is collinear to β_I^μ and k_2^μ is collinear to β_K^μ;
σ_7 ($\mathrm{C_1H}$),	if k_1^μ is collinear to β_I^μ and k_2^μ is hard;
σ_8 ($\mathrm{HC_2}$),	if k_1^μ is hard and k_2^μ is collinear to β_K^μ.

A two-loop example

$$
\begin{aligned}
\mathcal{N} = \Big\{ & \varnothing,\ \{\sigma_i\}_{i=1,\ldots,8}\,,\ \{\sigma_1,\sigma_i\}_{i=2,..,8}\,,\ \{\sigma_2,\sigma_4\},\ \{\sigma_2,\sigma_6\},\ \{\sigma_2,\sigma_7\},\ \{\sigma_2,\sigma_8\},\\
& \{\sigma_3,\sigma_5\},\ \{\sigma_3,\sigma_6\},\ \{\sigma_3,\sigma_7\},\ \{\sigma_3,\sigma_8\},\ \{\sigma_4,\sigma_7\},\ \{\sigma_5,\sigma_8\},\ \{\sigma_6,\sigma_7\},\ \{\sigma_6,\sigma_8\},\\
& \{\sigma_1,\sigma_2,\sigma_4\},\ \{\sigma_1,\sigma_2,\sigma_6\},\ \{\sigma_1,\sigma_2,\sigma_7\},\ \{\sigma_1,\sigma_2,\sigma_8\},\ \{\sigma_1,\sigma_3,\sigma_5\},\ \{\sigma_1,\sigma_3,\sigma_6\},\\
& \{\sigma_1,\sigma_3,\sigma_7\},\ \{\sigma_1,\sigma_3,\sigma_8\},\ \{\sigma_1,\sigma_4,\sigma_7\},\ \{\sigma_1,\sigma_5,\sigma_8\},\ \{\sigma_1,\sigma_6,\sigma_8\},\ \{\sigma_1,\sigma_7,\sigma_8\},\\
& \{\sigma_2,\sigma_4,\sigma_7\},\ \{\sigma_2,\sigma_6,\sigma_7\},\ \{\sigma_2,\sigma_6,\sigma_8\},\ \{\sigma_3,\sigma_5,\sigma_7\},\ \{\sigma_3,\sigma_5,\sigma_8\},\ \{\sigma_3,\sigma_6,\sigma_7\},\\
& \{\sigma_3,\sigma_6,\sigma_8\},\ \{\sigma_1,\sigma_2,\sigma_4,\sigma_7\},\ \{\sigma_1,\sigma_2,\sigma_6,\sigma_7\},\ \{\sigma_1,\sigma_2,\sigma_6,\sigma_8\},\ \{\sigma_1,\sigma_3,\sigma_5,\sigma_8\},\\
& \{\sigma_1,\sigma_3,\sigma_6,\sigma_7\},\ \{\sigma_1,\sigma_3,\sigma_6,\sigma_8\} \Big\}.
\end{aligned}
$$

There are 52 subtraction terms in the forest formula, and each term induces 8 different IR divergences, but they end up cancelling each other.

Comments on the IR cancellation

- There are various divergences in each term of the forest formula, but all these divergences form pairs to cancel each other.

- This IR cancellation is diagram-by-diagram.

- Nested divergences are cancelled directly. Overlapping divergences are cancelled with the help of enclosed pinch surfaces.
  - We proved that each enclosed pinch surface τ is a leading pinch surface of the original amplitude $\mathcal{A}$, so t_τ does appear in the forest formula.
  - In order to prove that $[t_\tau t_\sigma \mathcal{A} - t_\sigma \mathcal{A}]_{\rho^{\{\sigma\}}} = 0$, we checked that t_τ is exact at $\rho^{\{\sigma\}}$.
  - The analysis has also been generalized to repetitive approximations.

Outline

- Introduction
  - Infrared divergences in perturbative QCD
  - The forest structure of subtractions
- Approximations and induced IR divergences
  - Definition: hard-collinear & soft-collinear
  - Induced pinch surfaces
- A forest formula to subtract IR divergences
  - Expression
  - Pairwise IR cancellation
- Factorization --- a byproduct
  - Factorization near a pinch surface
  - Hard-soft-collinear factorization to all orders
- Summary and outlook

Factorization of $t_\sigma \mathcal{A}$

- **Near each pinch surface σ , by applying the approximations and the Ward identity,**

$$\langle M \,|T\left\{\partial_{\mu_1} A^{\mu_1} \partial_{\mu_2} A^{\mu_2} ... \partial_{\mu_n} A^{\mu_n}\right\}|\, N\rangle = 0.$$

the hard, jet and soft subdiagrams can be decoupled.

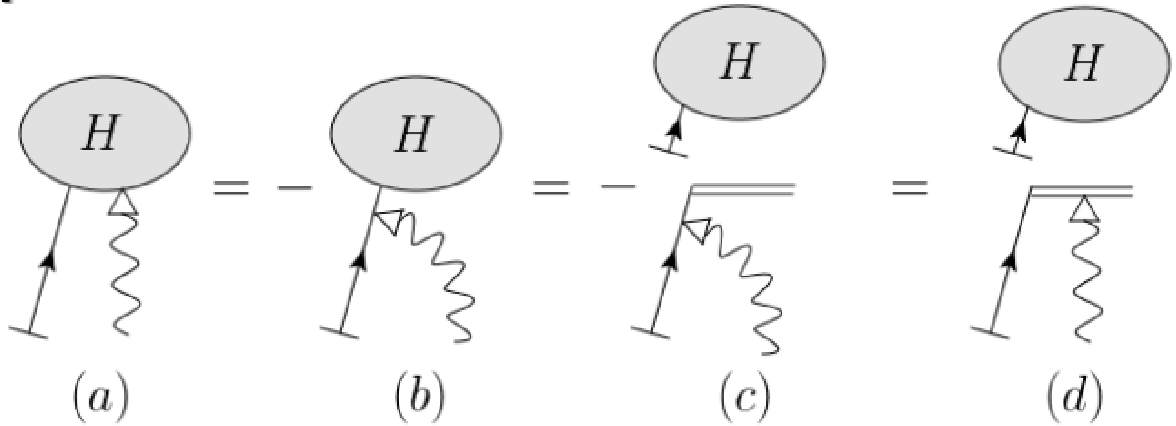
Factorization of $t_\sigma \mathcal{A}$

- **Near each pinch surface σ , by applying the approximations and the Ward identity,**

$$\langle M \left| T \left\{ \partial_{\mu_1} A^{\mu_1} \partial_{\mu_2} A^{\mu_2} ... \partial_{\mu_n} A^{\mu_n} \right\} \right| N \rangle = 0.$$

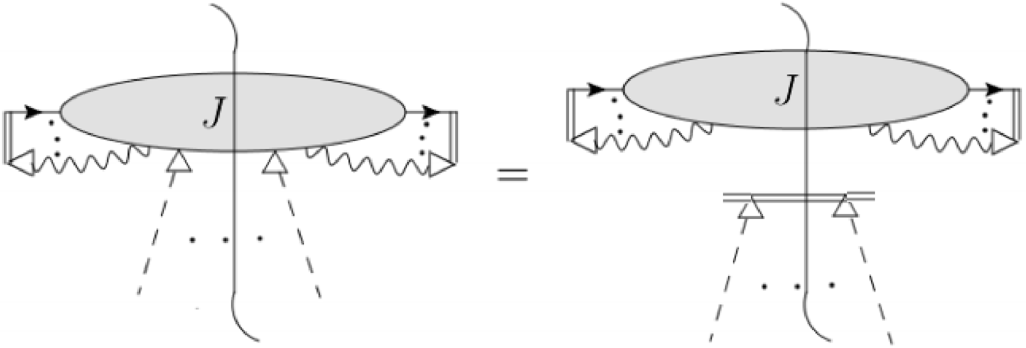
the hard, jet and soft subdiagrams can be decoupled.

- **To decouple J from H:**



Factorization of $t_\sigma \mathcal{A}$

- **To decouple S from J:**



- **Feynman rules of the Wilson lines**

$$= \frac{i}{p \cdot \beta + i\epsilon}$$

$$= ig\beta^\mu$$

Comments on the factorization above

- **We need to sum over different Feynman diagrams.**
- **Near each given σ, the sum over approximated amplitudes $\sum_{\mathcal{A}} t_\sigma \mathcal{A}$ can be written into a factorized form.**

  Can we factorize $\mathcal{A}$ without performing approximations?
- **Solution: the forest formula**

$$\mathcal{A}^{(n)} = -\sum_{\substack{F\in\mathcal{F}[\mathcal{A}^{(n)}]\\ F\neq\varnothing}} \prod_{\sigma\in F} (-t_\sigma)\, \mathcal{A}^{(n)} + R\left[\mathcal{A}^{(n)}\right].$$

To factorize $\mathcal{A}$ --- 5 steps

- **Final result:**

$$\mathcal{M} = \sum \mathcal{A} = \mathcal{H} \cdot \frac{\mathcal{J}_{\text{part}}}{\mathcal{J}_{\text{eik}}} \cdot \mathcal{S}$$

To factorize $\mathcal{A}$ --- 5 steps

- **Final result:**

$$\begin{aligned}\mathcal{M} = \sum \mathcal{A} &= \mathcal{H} \cdot \frac{\mathcal{J}_{\rm part}}{\mathcal{J}_{\rm eik}} \cdot \mathcal{S} \\ &= \mathcal{H} \cdot \frac{\mathcal{J}_{\rm part}}{(\mathcal{J}_{\rm eik})^{1/2}} \cdot \frac{\mathcal{S}}{(\mathcal{J}_{\rm eik})^{1/2}}\end{aligned}$$

To factorize $\mathcal{A}$ --- 5 steps

- **Final result:**

$$\mathcal{M} = \sum \mathcal{A} = \mathcal{H} \cdot \frac{\mathcal{J}_{\rm part}}{\mathcal{J}_{\rm eik}} \cdot \mathcal{S}$$

$$= \mathcal{H} \cdot \frac{\mathcal{J}_{\rm part}}{(\mathcal{J}_{\rm eik})^{1/2}} \cdot \frac{\mathcal{S}}{(\mathcal{J}_{\rm eik})^{1/2}}$$

- **Some related work:** **Sen (1981),**
  Catani (1998),
  Kidonakis, Oderda, Sterman (1998),
  Sterman, Tejeda-Yeomans (2003),
  Feige, Schwartz (2014),
  Erdogan, Sterman (2015).

Outline

- Introduction
  - ▷ Infrared divergences in perturbative QCD
  - ▷ The forest structure of subtractions
- Approximations and induced IR divergences
  - ▷ Definition: hard-collinear & soft-collinear
  - ▷ Induced pinch surfaces
- A forest formula to subtract IR divergences
  - ▷ Expression
  - ▷ Pairwise IR cancellation
- Factorization --- a byproduct
  - ▷ Factorization near a pinch surface
  - ▷ Hard-soft-collinear factorization to all orders
- Summary and outlook

Summary

- We first briefly reviewed the infrared structure in gauge theories. Each IR divergent region is called a pinch surface, which can be identified by hard-collinear and soft-collinear approximations.

- We then studied the pinch surfaces of the approximated amplitudes, which are generated by hard-collinear and soft-collinear approximations. They are very different from those of the original amplitude.

- We then studied the pairwise cancellations of the divergences in the forest formula: nested divergence and overlapping divergence. We developed the concept of enclosed pinch surface for the latter.

- The forest formula we derived can be applied to show the hard-collinear-soft factorization of scattering amplitudes.

Outlook

- Our study on the induced pinch surfaces can be applied to the study of the Soft-Collinear Effective Theory (SCET).

- The forest-based structure has mathematical interpretations, for example, the Hopf algebra.

- This project focuses on wide-angle scattering. It should be extended to other QCD processes, like the Regge limit. But then we need to deal with the Glauber region.

- This project is only for amplitude. We should generalize the analysis to (weighted) cross sections, and we are on the way!

谢谢！